Final Examination, SAMPLE ONLY

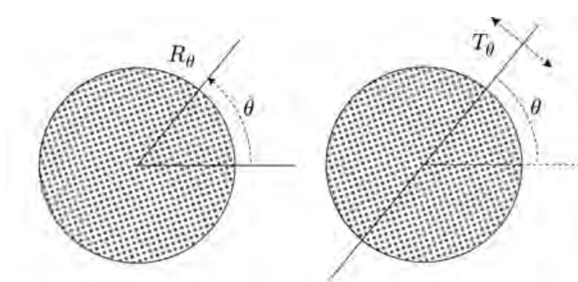
December, 2023

This sample exam is intended to resemble the Final Examination (8:00–10:00 am on Friday, December 15, 2023 in our usual lecture room, BU 209; we are also discussing an alternative time later that day) in approximate length, difficulty, and style, although clearly the content may differ. The actual content will consist of all material covered in class this semester, and all related handouts. Somewhat greater weight will be placed on the later material (covered after midterm Test).

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an 8.5″×11″ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Time permitted: 120 minutes. Total value of questions: 100 points (plus 14 bonus points).

1. *(12 points)* Consider the symmetric group $G = S_7$ and the centralizer of a transposition
$$H = C_G\big((67)\big) = \{\sigma \in G \,:\, \sigma(67) = (67)\sigma\}.$$
   (a) Determine $|H|$.
   (b) Identify H (up to isomorphism) using notation for the groups we have studied in class this semester. Justify your answer.

2. *(12 points)* Consider the closed disk $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$, and let G be the symmetry group of D. For each angle θ, denote by $R_\theta \in G$ the rotational symmetry of D by an angle θ conterclockwise about its center. Also denote by $T_\theta \in G$ the reflective symmetry of D whose axis has angle θ relative to x-axis as shown.



   (a) Show that every rotational symmetry $R_\theta \in G$ is a product of two reflective symmetries, i.e. $R_\theta = T_\alpha T_\beta$ for some angles α, β depending on θ.
   (b) Show that any two reflective symmetries $T_\alpha, T_\beta \in G$ are conjugate in G; in fact, show that $T_\beta = R_\theta T_\alpha R_\theta^{-1}$ for some angle θ depending on α and β.

3. *(12 points)* Let $G = GL_2(\mathbb{R})$ and define $\theta : GL_2(\mathbb{R}) \to GL_2(\mathbb{R})$ by $\theta(A) = \det(A)A$ for every matrix $A \in G$. (For example, $\theta(\left(\begin{smallmatrix} 1 & 2 \\ 0 & 3 \end{smallmatrix}\right)) = 3\left(\begin{smallmatrix} 1 & 2 \\ 0 & 3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 3 & 6 \\ 0 & 9 \end{smallmatrix}\right)$.) Is θ an automorphism of G? Justify your answer.

4. *(12 points)* Let H and K be subgroups of a group G. Assume K is a *normal* subgroup of G. Must $H \cap K$ be a normal subgroup (a) of H? (b) of K? (c) of G? Justify your answers.

5. *(12 points)* Let $G = GL_3(\mathbb{R})$, the group of all invertible real 3×3 matrices.
   (a) Does G have a subgroup isomorphic to $GL_2(\mathbb{R})$? Justify your answer.
   (b) Does G have a subgroup of order 5? Justify your answer.
   (c) Does G have an infinite abelian subgroup? Justify your answer.

6. *(12 points)* Let $G = GL_2(\mathbb{F}_5)$, the group of all invertible real 2×2 matrices over the field $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ of order 5. Note that G permutes the 25 vectors $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathbb{F}_5^2$. Find the stabilizer $\mathrm{Stab}_G(\mathbf{u})$ of the vector $\mathbf{u} = \left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$. What is its order?

7. *(12 points)* In A_6, how many elements of order 4 are there? How many conjugacy classes of such elements are there? Do the same for elements of order 5.

8. *(30 points)* Answer TRUE or FALSE to each of the following statements.
   (a) Every finite group is isomorphic to a subgroup of S_n for some $n \geq 1$.
   ______*(True/False)*

   (b) Every nontrivial finite group contains an element of prime order.
   ______*(True/False)*

   (c) If two elements x, y in a group G satisfy $(xy)^6 = 1$, then necessarily $(yx)^6 = 1$.
   ______*(True/False)*

   (d) A group homomorphism $\theta : G \to H$ is one-to-one, iff its kernel is trivial.
   ______*(True/False)*

   (e) If G is an abelian group, then any two automorphisms of G commute with each other.
   ______*(True/False)*

   (f) If a finite group G acts transitively on a set X (i.e. there is only one orbit), then for all $x, y \in X$, the stabilizers G_x and G_y necessarily have the same order.
   ______*(True/False)*

   (g) If H and K are subgroups of a group G, then so is their product $HK = \{hk : h \in H,\ k \in K\}$.
   ______*(True/False)*

(h) If g, h are elements of a group G, then there exists a homomorphism $\phi : G \to G$ such that $\phi(g) = h$. ______*(True/False)*

(i) If G is a group of order n and $\gcd(k, n) = 1$, then the k-th power $\theta_k : G \to G$, $\theta_k(x) = x^k$ is necessarily bijective. ______*(True/False)*

(j) If G and H are finite groups having the same number of elements of each order, then necessarily G and H are isomorphic. ______*(True/False)*