Sample Test

October, 2023

This sample test is intended to resemble the Test (Monday, October 30, 2023 during class time) in approximate length, difficulty, and style, although clearly the content may differ. The actual content will consist of all material covered in class prior to the test.

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an $8.5'' \times 11''$ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Time permitted: 50 minutes.

1. *(15 points)* Find a subgroup $G \leqslant S_4$ isomorphic to the dihedral group of order 8. Answer by exhibiting G as a set of 8 explicit permutations (a Cayley table is not necessary).

2. *(20 points)* Let S be the set of all 2×2 real matrices of the form

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where $a \in \mathbb{R}$. Is every invertible 2×2 real matrix expressible as a product of elements of S? Justify your answer.

3. *(20 points)* Let G be a multiplicative group containing elements g, h of order $|g| = 3$ and $|h|=5$.
   (a) What can be said about $|gh|$ based on the information given?
   (b) What can be said about $|gh|$ if g commutes with h?

4. *(15 points)* Let H_1, H_2 be subgroups of a multiplicative group G. Give an example to show that the subset defined by

$$H_1H_2 = \{h_1h_2 \,:\, h_1 \in H_1,\ h_2 \in H_2\} \subseteq G$$

is not necessarily a subgroup.

5. *(30 points)* Answer TRUE or FALSE to each of the following statements.

(a) For every positive integer n, there is a group of order n. _____*(True/False)*

(b) If two groups are isomorphic, then they necessarily have the same order. _____*(True/False)*

(c) There exist infinitely many groups of order 24, no two of which are isomorphic. _____*(True/False)*

(d) If G is a finite group, then every element of G has finite order. _____*(True/False)*

(e) There exist physical objects with rotational symmetry but no reflective symmetry. _____*(True/False)*

(f) The multiplicative group $\mathbb{C}^{\times}$ of nonzero complex numbers has elements of all possible orders, finite and infinite. _____*(True/False)*

(g) The additive group of integers $\mathbb{Z}$ is isomorphic to the additive group of even integers $2\mathbb{Z} = \{2k : k \in \mathbb{Z}\}$. _____*(True/False)*

(h) If G is a group containing two elements g, h of finite order, then their product gh has finite order. _____*(True/False)*

(i) Let G be a set with two binary operations '$*$' and '$\circ$', such that $g \circ f = f * g$ for all $f, g \in G$. If G is a group under the operation '$*$', then G is also a group under the operation '$\circ$'. _____*(True/False)*

(j) Given a finite group G of order n, there is a binary operation '$*$' on the set $[n] = \{1, 2, \ldots, n\}$ which defines a group isomorphic to G. _____*(True/False)*