HW4

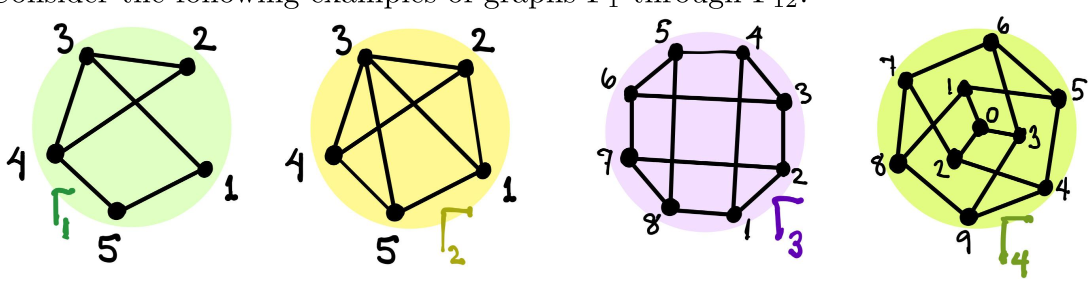
Due 5:00 pm, Friday, May 8, 2026 on WyoCourses.

Instructions: See the syllabus for general instructions for completing homework. Further details are found at the FAQ page linked from the syllabus. Always check your answers wherever feasible. Write clearly, using correct notation. Some of these questions are open-ended. *I sincerely hope you have fun with this!*

1. *(20 points)* Consider the functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{x-1}{x}$, and let G be the group generated by $f(x)$ and $g(x)$ under the binary operation of composition of functions. Determine the order of G, and count elements of each order in G. (*Remember to always simplify!*) Is G isomorphic to any group that we have studied this semester? Explain.

A *graph* has a set of *vertices* (often represented as dots in a picture of the graph) and *edges* which represent adjacent pairs of vertices. We often *label* the vertices of the graph $1, 2, \ldots, n$ (and n is called the *order* of the graph). We will discuss graphs more extensively in class; but here are a few examples to help jog your memory. Given vertices x and y, we write $x \sim y$ or $x \not\sim y$ according as x and y are adjacent or not; in other words, according as there is an edge between x and y or not. The *automorphism group* of a graph Γ, denoted $\operatorname{Aut}\Gamma$, is the set of all permutations of the vertices which preserve the adjacency relation; in other words, the set of all permutations σ of the vertices such that $x \sim y$ iff $\sigma(x) \sim \sigma(y)$.

Consider the following examples of graphs Γ_1 through Γ_{12}.

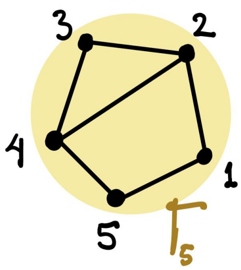


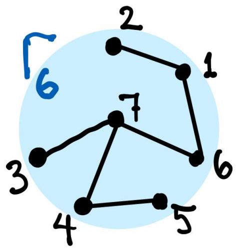
Γ_1 is a graph of order 5 (it has 5 vertices). This graph has 6 edges $\{1,3\}$, $\{1,5\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$, $\{4,5\}$. Its automorphism group is $\operatorname{Aut}\Gamma_1 = \langle(15)(34)\rangle$ of order 2.

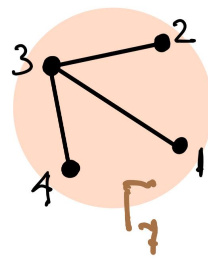
Γ_2 is a graph of order 5 with automorphism group $\operatorname{Aut}\Gamma_2 = \langle(14),(1245)\rangle$, a dihedral group of order 8.

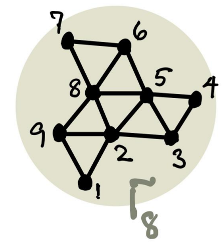
Γ_3 is a graph of order 8 with automorphism group $\operatorname{Aut}\Gamma_3 = \langle(1234)(5876),(248)(357),$ $(16)(25)(38)(47)\rangle$, a group of order 48 isomorphic to the isometry group of a cube. Our vertex labelling $1, 2, \ldots, 8$ agrees with the labelling used for the cube in HW2 (as you can see from our choice of Hamilton circuit).

Γ_4 is a graph of order 10 with automorphism group $\operatorname{Aut}\Gamma_4 = \langle(09234)(18765), (47)(58)(69)\rangle$, a group of order 120 isomorphic to S_5. This is the celebrated *Petersen graph*. I am indexing vertices $0, 1, 2, \ldots, 9$ rather than $1, 2, \ldots, 10$ just to avoid double digits and commas in my cycle notation for permutations.







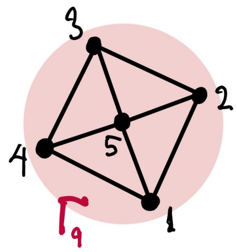


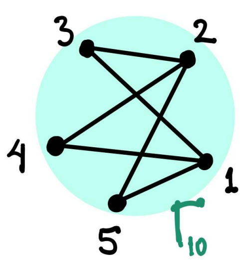
Γ_5 is a graph of order 5 with automorphism group $\operatorname{Aut}\Gamma_5 = \langle(15)(24)\rangle$ of order 2. Note that $\Gamma_5 \cong \Gamma_1$ as seen by relabelling of the vertices via the transposition (23). Furthermore, the two groups $\operatorname{Aut}\Gamma_5$ and $\operatorname{Aut}\Gamma_1$ are conjugate in S_5 (again using conjugation by (23)).

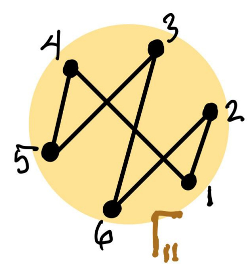
Γ_6 is a graph of order 7 with trivial automorphism group $\operatorname{Aut}\Gamma_6 = \langle()\rangle$. This being a trivial group, one often abuses notation by writing simply $\operatorname{Aut}\Gamma_6 = 1$.

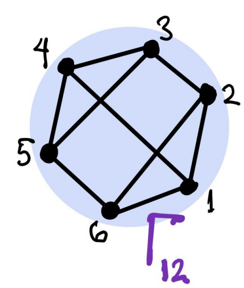
Γ_7 is a graph of order 4 with automorphism group $\operatorname{Aut}\Gamma_7 = \langle(12), (14)\rangle \cong S_3$ of order 6.

Γ_8 is a graph of order 9 with automorphism group $\operatorname{Aut}\Gamma_8 = \langle(147)(258)(369)\rangle$ of order 3. Examples of graphs with automorphism group of order 3 are harder to find than graphs with automorphism group isomorphic to S_3.









Γ_9 is a graph of order 5 with automorphism group $\operatorname{Aut}\Gamma_9 = \langle(1234), (13)\rangle$, a dihedral group of order 8.

Γ_{10} is a graph of order 5 with automorphism group $\operatorname{Aut}\Gamma_{10} = \langle(12), (34), (45)\rangle \cong S_2 \times S_3 \cong C_2 \times S_3$ of order 12. (Here C_2 is cyclic of order 2.)

2. *(20 points)* Verify that $\Gamma_2 \cong \Gamma_9$ by finding an explicit permutation $\tau \in S_5$ that maps Γ_2 to Γ_9. If $\phi : S_5 \to S_5$ is the inner automorphism $\phi = \phi_5$ mapping $\sigma \mapsto \tau\sigma\tau^{-1}$, verify that ϕ maps $\operatorname{Aut}\Gamma_2$ to $\operatorname{Aut}\Gamma_9$, so that these two subgroups of S_5 are conjugate in S_5. (You should compare with our example $\Gamma_1 \cong \Gamma_5$.)

3. *(20 points)* Determine the automorphism group of Γ_{11}. Specify $\mathrm{Aut}\,\Gamma_{11}$ as a permutation group generated by at most two elements of S_6, and find its order.

4. *(20 points)* Determine $\mathrm{Aut}\,\Gamma_{12}$. Specify $\mathrm{Aut}\,\Gamma_{12}$ as a permutation group generated by at most four elements of S_6, and find its order.

5. *(20 points)* Consider the group $G = C_2 \times C_4 \times C_8$ of order 64. (Here C_n is a cyclic group of order n.) How many elements of each order does G have? (Be sure to check that the total number of elements adds up correctly.)

The group $G = S_5 \times S_5$ of order 14400 has *many* subgroups isomorphic to S_5. If we consider

$$G = \{(\alpha, \beta) : \alpha, \beta \in S_5\},$$

then G has subgroups

$$X = S_5 \times 1 = \{(\alpha, ()) : \alpha \in S_5\} \cong S_5; \quad \text{and}$$
$$Y = 1 \times S_5 = \{((), \beta) : \beta \in S_5\} \cong S_5\,.$$

Of course every element of G is uniquely expressible as an element of X times an element of Y. There is also a 'diagonal' subgroup $D < G$ of order 120 isomorphic to S_5 given by

$$D = \{(\alpha, \alpha) : \alpha \in S_5\}.$$

But this *barely scratches the surface*!

6. *(20 points)* Let $G = S_5 \times S_5$, and let ϕ be any automorphism of S_5.
   (a) Show that $\{(\alpha, \phi(\alpha)) : \alpha \in S_5\}$ is a subgroup of G of order 120 isomorphic to S_5.
   (b) In (a), does ϕ really need to be an automorphism? Can it be just any function $\phi : S_5 \to S_5$, or how special does it have to be?
   (c) What lower estimate can you find for the number of subgroups of G isomorphic to S_5? E.g. at least a hundred? At least a thousand?

Likewise, the group S_{10} has *many* subgroups isomorphic to the group $G = S_5 \times S_5$. For example, the set of all $\sigma \in S_{10}$ such that σ preserves $\{1, 2, 3, 4, 5\}$ (i.e. $\sigma(i) \in \{1, 2, 3, 4, 5\}$ whenever $i \in \{1, 2, 3, 4, 5\}$; and $\sigma(i) \in \{6, 7, 8, 9, 10\}$ whenever $i \in \{6, 7, 8, 9, 10\}$) is a subgroup of S_{10} isomorphic to G. But there are many more such subgroups.

7. *(20 points)* Show that S_{10} has *many* subgroups isomorphic to $G = S_5 \times S_5$. How many can you find? At least 100? At least 1000?

By the way, S_{10} has 79,632 subgroups isomorphic to S_5 (if I did the count correctly, at least it looks right). This number won't help you at all with #6,7, however, as it includes more than just the subgroups related to $S_5 \times S_5$ (it includes all subgroups conjugate to $\mathrm{Aut}\,\Gamma_4 \cong S_5$, plus some subgroups inspired by #1, perhaps surprisingly). All this and much much more awaits you if you decide to go further into group theory.