Solutions to HW3

December, 2023

1. (a) $G = \{\iota, \sigma, \tau, \sigma\tau, \tau\sigma, \sigma\tau\sigma\}$ where

$$\iota(x) = x; \quad \sigma(x) = \tfrac{1}{x}; \quad \tau(x) = 1 - x; \quad (\sigma\tau)(x) = \tfrac{1}{1-x};$$

$$(\tau\sigma)(x) = 1 - \tfrac{1}{x}; \quad (\sigma\tau\sigma)(x) = (\tau\sigma\tau)(x) = \tfrac{x}{x-1}\,.$$

(b) $|G| = 6$

(c) G has one element of order 1 (the identity, ι); three elements of order 2 ($\sigma, \tau, \sigma\tau\sigma$) and two elements of order 3 ($\sigma\tau$ and $\tau\sigma$).

(d) An explicit isomorphism $G \cong S_3$ is shown by Cayley tables:

G	ι	τ	σ	$\sigma\tau\sigma$	$\sigma\tau$	$\tau\sigma$
ι	ι	τ	σ	$\sigma\tau\sigma$	$\sigma\tau$	$\tau\sigma$
τ	τ	ι	$\tau\sigma$	$\sigma\tau$	$\sigma\tau\sigma$	σ
σ	σ	$\sigma\tau$	ι	$\tau\sigma$	τ	$\sigma\tau\sigma$
$\sigma\tau\sigma$	$\sigma\tau\sigma$	$\tau\sigma$	$\sigma\tau$	ι	σ	τ
$\sigma\tau$	$\sigma\tau$	σ	$\sigma\tau\sigma$	τ	$\tau\sigma$	ι
$\tau\sigma$	$\tau\sigma$	$\sigma\tau\sigma$	τ	σ	ι	$\sigma\tau$

S_3	()	(12)	(13)	(23)	(123)	(132)
()	()	(12)	(13)	(23)	(123)	(132)
(12)	(12)	()	(132)	(123)	(23)	(13)
(13)	(13)	(123)	()	(132)	(12)	(23)
(23)	(23)	(132)	(123)	()	(13)	(12)
(123)	(123)	(13)	(23)	(12)	(132)	()
(132)	(132)	(23)	(12)	(13)	()	(123)

Alternatively, G permutes $\{0, 1, \infty\}$ inducing all six permutations of this set as

$$\iota = (); \quad \sigma = (0, \infty); \quad \tau = (0, 1); \quad \sigma\tau = (0, 1, \infty);$$

$$\tau\sigma = (0, \infty, 1); \quad \sigma\tau\sigma = \tau\sigma\tau = (1, \infty)$$

and so relabeling the three points $0, 1, \infty$ as $1, 2, 3$ (respectively) gives the isomorphism $G \cong S_3$ exhibited above.

2. (a) $\gamma = \alpha\beta^{-1} = \alpha\beta^3$

(b) $|G| = 120$ as found by Maple.

(c) G has one element of order 1; 25 elements of order 2; 20 elements of order 3; 30 elements of order 4; 24 elements of order 5; and 20 elements of order 6, as found by counting elements of G as listed by Maple.

(d) G contains 60 even permutations and 60 odd permutations, again found by counting elements of G as listed by Maple.

(e) The characteristics of G listed above agree with those of S_5, so it is natural to conjecture that $G \cong S_5$.

(f) The group $SL_2(\mathbb{F}_5)$ has order 120 but is not isomorphic to G (since $SL_2(\mathbb{F}_5)$ has a center of order 2 whereas G has trivial center). A third group of order 120, not isomorphic to either $SL_2(\mathbb{F}_5)$, is the direct product $A_5 \times C_2$ where C_2 is cyclic of order 2. The latter group $A_5 \times C_2$ also has a center of order 2; and it is isomorphic to the symmetry group of a regular dodecahedron (or the regular icosahedron).

3. (a) $|G| = 480$ since in choosing an element $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in G$, there are $5^2 - 1 = 24$ choices for the first column $\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)$ (any nonzero vector), and then $5^2 - 5 = 20$ choices for the second column $\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)$ (any vector not a scalar multiple of the first column); thus $24{\cdot}20 = 480$ elements of G in all.

   (b) $Z(G) = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) : 0 \neq a \in \mathbb{F}_5\right\}$ is a cyclic subgroup of order 4 generated by $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$.

   (c) In the natural action of G on the two-dimensional vector space $\mathbb{F}_5^2$, the stabilizer of the zero vector $\mathbf{0} = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ is $G_{\mathbf{0}} = G$.

   (d) $C_G(g) = \left\{\left(\begin{smallmatrix} a & 0 \\ c & a \end{smallmatrix}\right) : a, c \in \mathbb{F}_5,\ a \neq 0\right\}$, of order $|C_G(g)| = 20$.

   (e) Conjugates of g in G must have trace 2 and determinant 1; and of course we must exclude the identity matrix. There are only 24 such elements in G, namely $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 2 & -\frac{1}{c} \\ c & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -\frac{1}{c} \\ c & 2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 3 & \frac{1}{c} \\ c & 4 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 4 & \frac{1}{c} \\ c & 3 \end{smallmatrix}\right)$ where $0 \neq c \in \mathbb{F}_5$. Since we require $[G : C_G(g)] = \frac{480}{20} = 24$ conjugates, these must in fact be all the conjugates of g in G.

4. (a) Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} r & t \\ s & u \end{smallmatrix}\right]$ in G; then

$$f_A(f_B(x)) = \frac{a\left(\frac{rx+s}{tx+u}\right) + b}{c\left(\frac{rx+s}{tx+u}\right) + d} = \frac{a(rx+s) + b(tx+u)}{c(rx+s) + d(tx+u)} = \frac{(ar+bt)x + (as+bu)}{(cr+dt)x + (cs+du)} = f_{AB}(x)$$

   since

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} ar+bt & as+bu \\ cr+dt & cs+du \end{bmatrix}.$$

   (b) Since the map $GL_2(\mathbb{F}_5) \to PGL_2(\mathbb{F}_5)$ is surjective and 4-to-1, $|PGL_2(\mathbb{F}_5)| = \frac{480}{4} = 120$.

   (c) We find that $\alpha : x \mapsto 3x+1$ and $\beta : x \mapsto \frac{x}{x+3}$, in each case by solving a system of six equations for the unkown coefficients in each fractional linear transformation; thus we may take $A = \left[\begin{smallmatrix} 3 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 3 \end{smallmatrix}\right]$. (As noted in the assignment, however, A and B may be replaced by any nonzero scalar multiples thereof.)

   (d) Since $\langle \alpha, \beta \rangle = \langle f_A, f_B \rangle \leqslant PGL_2(\mathbb{F}_5)$ where both $PGL_2(\mathbb{F}_5)$ and the subgroup $\langle \alpha, \beta \rangle$ have order 120, equality must hold: $|PGL_2(\mathbb{F}_5)| = 120$ as claimed.

HW3 #2. I will represent the points 1,2,3,4,0,∞ by 1,2,3,4,5,6 in the context of this Maple worksheet.

```
> with(GroupTheory):
> alpha:=Perm([[5,1,4,3]]); beta:=Perm([[1,4,2,6]]);
```

$$\alpha := (1, 4, 3, 5)$$

$$\beta := (1, 4, 2, 6) \quad (1)$$

```
> local gamma; gamma:=Perm([[5,1,6,2,3]]);
```

$$\gamma := (1, 6, 2, 3, 5) \quad (2)$$

The following product uses left-to-right composition:

```
> beta.beta.beta.alpha;
```

$$(1, 6, 2, 3, 5) \quad (3)$$

```
> G:=PermutationGroup(alpha,beta);
```

$$G := \langle (1, 4, 3, 5), (1, 4, 2, 6) \rangle \quad (4)$$

```
> GroupOrder(G);
```

$$120 \quad (5)$$

Count the number of elements of each order in *G*. You can list all elements and count manually, but I will use Maple to count for me. I could program this in Maple from first principles, but instead let me look up suitable builtin commands using 'Help'.

```
> E:=Elements(G);
```

$E :=$ {( ), (1, 2, 3, 6), (1, 2, 4, 3), (1, 2, 5, 4), (1, 2, 6, 5), (1, 3, 2, 5), (1, 3, 4, 2), (1, 3, 5, 6), (1, 3, 6, 4), (1, 4, 2, 6), (1, 4, 3, 5), (1, 4, 5, 2), (1, 4, 6, 3), (1, 5, 2, 3), (1, 5, 3, 4), (1, 5, 4, 6), (1, 5, 6, 2), (1, 6, 2, 4), (1, 6, 3, 2), (1, 6, 4, 5), (1, 6, 5, 3), (2, 3, 4, 6), (2, 3, 5, 4), (2, 4, 5, 3), (2, 4, 6, 5), (2, 5, 3, 6), (2, 5, 6, 4), (2, 6, 3, 5), (2, 6, 4, 3), (3, 4, 5, 6), (3, 6, 5, 4), (1, 2, 3, 4, 5), (1, 2, 4, 5, 6), (1, 2, 5, 6, 3), (1, 2, 6, 3, 4), (1, 3, 2, 4, 6), (1, 3, 4, 6, 5), (1, 3, 5, 2, 4), (1, 3, 6, 5, 2), (1, 4, 2, 5, 3), (1, 4, 3, 6, 2), (1, 4, 5, 3, 6), (1, 4, 6, 2, 5), (1, 5, 2, 6, 4), (1, 5, 3, 2, 6), (1, 5, 4, 3, 2), (1, 5, 6, 4, 3), (1, 6, 2, 3, 5), (1, 6, 3, 5, 4), (1, 6, 4, 2, 3), (1, 6, 5, 4, 2), (2, 3, 6, 4, 5), (2, 4, 3, 5, 6), (2, 5, 4, 6, 3), (2, 6, 5, 3, 4), (1, 2, 3, 5, 6, 4), (1, 2, 4, 6, 3, 5), (1, 2, 5, 3, 4, 6), (1, 2, 6, 4, 5, 3), (1, 3, 2, 6, 5, 4), (1, 3, 4, 5, 2, 6), (1, 3, 5, 4, 6, 2), (1, 3, 6, 2, 4, 5), (1, 4, 2, 3, 6, 5), (1, 4, 3, 2, 5, 6), (1, 4, 5, 6, 2, 3), (1, 4, 6, 5, 3, 2), (1, 5, 2, 4, 3, 6), (1, 5, 3, 6, 4, 2), (1, 5, 4, 2, 6, 3), (1, 5, 6, 3, 2, 4), (1, 6, 2, 5, 4, 3), (1, 6, 3, 4, 2, 5), (1, 6, 4, 3, 5, 2), (1, 6, 5, 2, 3, 4), (1, 2)(3, 5), (1, 2)(4, 6), (1, 3)(2, 6), (1, 3)(4, 5), (1, 4)(2, 3), (1, 4)(5, 6), (1, 5)(2, 4), (1, 5)(3, 6), (1, 6)(2, 5), (1, 6)(3, 4), (2, 3)(5, 6), (2, 4)(3, 6), (2, 5)(3, 4), (2, 6)(4, 5), (3, 5)(4, 6), (1, 2, 3)(4, 6, 5), (1, 2, 4)(3, 6, 5), (1, 2, 5)(3, 6, 4), (1, 2, 6)(3, 5, 4), (1, 3, 2)(4, 5, 6), (1, 3, 4)(2, 5, 6), (1, 3, 5)(2, 6, 4), (1, 3, 6)(2, 5, 4), (1, 4, 2)(3, 5, 6), (1, 4, 3)(2, 6, 5), (1, 4, 5)(2, 6, 3), (1, 4, 6)(2, 3, 5), (1, 5, 2)(3, 4, 6), (1, 5, 3)(2, 4, 6), (1, 5, 4)(2, 3, 6), (1, 5, 6)(2, 3, 4), (1, 6, 2)(3, 4, 5), (1, 6, 3)(2, 4, 5), (1, 6, 4)(2, 5, 3), (1, 6, 5)(2, 4, 3), (1, 2)(3, 4)(5, 6), (1, 2)(3, 6)(4, 5), (1, 3)(2, 4)(5, 6), (1, 3)(2, 5)(4, 6), (1, 4)(2, 5)(3, 6), (1, 4)(2, 6)(3, 5), (1, 5)(2, 3)(4, 6), (1, 5)(2, 6)(3, 4), (1, 6)(2, 3)(4, 5), (1, 6)(2, 4)(3, 5)} **(6)**

```
> element_orders:=[seq(PermOrder(g),g in E)];
```

element_orders := [3, 4, 6, 3, 2, 5, 4, 3, 4, 5, 4, 2, 2, 2, 6, 2, 4, 5, 4, 3, 5, 6, 3, 6, 3, 4, 5, 6, 5, 4, 3, 4, 4, 2, 2, 4, 5, 4, 4, 5, 4, 2, 4, 5, 2, 5, 4, 2, 4, 4, 2, 3, 5, 1, 4, 4, 3, 3, 2, 5, 4, 5, 4, 2, 2, 5, 6, 3, 6, 2, 3, 6, 4, 2, 5, 6, 5, 6, 3, 4, 3, 2, 3, 6, 5, 6, 4, 2, 6, 2, 3, 5, 4, 2, 3, 4, 3, 5, 4, 6, 2, 6, 3, 5, 6, 5, 6, 5, 2, 2, 4, 6, 4, 2, 3, 6, 2, 5, 6, 5] **(7)**

```
> with(ListTools): Collect(element_orders);
```

$$[[1, 1], [2, 25], [3, 20], [4, 30], [5, 24], [6, 20]] \quad (8)$$

Verify that *G* has trivial center.

```
> Center(G); GroupOrder(%);
```

$$Z(\langle (1, 4, 3, 5), (1, 4, 2, 6) \rangle)$$

$$1 \quad (9)$$