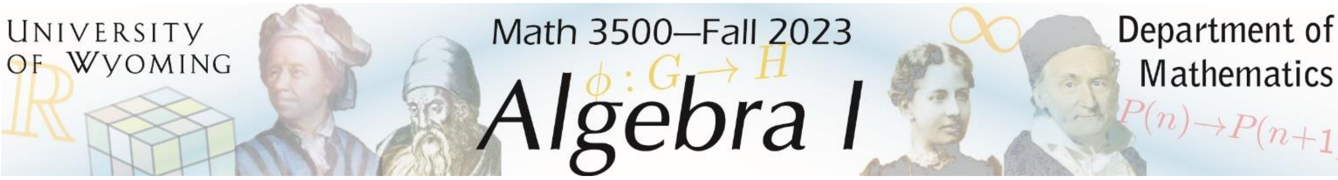

HW3

Due 5:00pm Wednesday, December 6, 2023 on WyoCourses

Instructions: See the syllabus for general instructions for completing homework. Further details are found at the FAQ page linked from the syllabus. Always check your answers wherever feasible. Write clearly, using correct notation. Total value of questions: 100 points.

1. (*20 points*) Let $X = \mathbb{F}_2 \cup \{\infty\} = \{0,1,\infty\}$ and consider the two permutations σ, τ of X given by

$$\sigma(x) = \tfrac{1}{x}; \qquad \tau(x) = 1 - x.$$

Note that $\sigma : X \to X$ is bijective, with the convention that $\sigma(0) = \frac{1}{0} = \infty$ and $\sigma(\infty) = \frac{1}{\infty} = 0$; similarly $\tau : X \to X$ is bijective with the understanding that $\tau(\infty) = 1 - \infty = \infty$. (In this context there is no distinction between ∞ and $-\infty$.) Let $G = \langle \sigma, \tau \rangle$, the subgroup of $Sym(X)$ generated by σ and τ under composition, i.e. the group of all permutations that can be written by composing σ's and τ's any number of times, and in any order.

   a. List all elements of G. Simplify these elements as much as possible.
   b. What is the order of G?
   c. How many elements of each order does G have?
   d. Find an explicit isomorphism between G and a group studied previously this semester.

2. (*30 points*) In the puzzle shown, six round tiles (labelled $0,1,2,3,4,\infty$) are moved around tracks. The group G, consisting of all legal moves, is generated by the two 4-cycles $\alpha = (0,1,4,3)$ and $\beta = (1,4,2,\infty)$. Note that G is a subgroup of $Sym\{0,1,2,3,4,\infty\}$, the group of all permutations of the six tiles, a group isomorphic to S_6.

   a. The 5-cycle around the outer track is $\gamma = (0,1,\infty,2,3)$. Express γ as a combination of the generators α and β, i.e. a product of powers of the two generators. (This shows that $\gamma \in G = \langle \alpha, \beta \rangle$.)
   b. Determine the order of G.
   c. How many elements of each order does G have?
   d. How many elements of G are even permutations, and how many are odd permutations?
   e. The group G is isomorphic to a group studied previously this semester. Based on your answers to (b) and (c) above, *conjecture* which group this is.
   f. Name another group studied previously this semester, having the same order as G, but which is not isomorphic to G. The information in (c) can be used to justify why these two groups are in fact not isomorphic.

You may use Maple or other comparable software in answering this problem. Before starting, you will probably want to temporarily rename the tiles 1,2,3,4,5,6 to be acceptable for input to Maple. To answer (c) and (d), it is helpful first to list all elements of G, use the Maple command `Elements(G);`

3. (*30 points*) Let $\mathbb{F}_5 = \{0,1,2,3,4\}$ be the field of order 5 (the integers mod 5). We denote by $GL_2(\mathbb{F}_5)$ the group of all invertible 2×2 matrices over $\mathbb{F}_5$; thus

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{F}_5, ad - bc \neq 0 \right\}.$$

   a. What is the order of G?
   b. Explicitly describe the center $Z(G)$, in particular giving its order. Is this subgroup cyclic? If so, give a generator for the center.
   c. Explicitly describe G_0 , the stabilizer of the point 0 in G.
   d. Explicitly describe the centralizer $C_G(g)$ of the element $g \in G$ defined by $g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. In particular, you should indicate the order $|C_G(g)|$.
   e. How many conjugates does g have in G ? List them.

4. (*20 points*) Let $\mathbb{F}_5 = \{0,1,2,3,4\}$ be the field of order 5, as in #3 (also known as the ring of integers mod 5) and let $X = \mathbb{F}_5 \cup \{\infty\} = \{0,1,2,3,4,\infty\}$. A *fractional linear transformation* on X is a map $f : X \to X$ of the form

$$f(x) = \frac{ax+b}{cx+d}$$

where $a,b,c,d \in \mathbb{F}_5$ such that $ad - bc \neq 0$. (The condition $ad - bc \neq 0$ is required in order for f to be a well-defined bijection.) As in #1, f is well-defined provided we adopt suitable conventions regarding ∞; in particular for the function f given above,

$$f(\infty) = \begin{cases} \frac{a}{c}, & if\ c \neq 0; \\ \infty, & if\ c = 0 \end{cases}$$

just as one would expect from taking limits. We also have $\frac{1}{0} = \infty$, $\infty + 1 = \infty$, etc. as in #1. There is no danger of encountering an ill-defined expression like $\frac{0}{0}$ because of our requirement that $ad - bc \neq 0$. For example, the fractional linear transformation

$$f(x) = \frac{2x+3}{3x+4}$$

acts on X as the permutation $(0,2,\infty,4,1)$, i.e. $f(0) = 2$, $f(1) = 0$, $f(2) = \infty$, … , $f(\infty) = 4$. Let us clarify the connection between invertible 2×2 matrices and fractional linear transformations: Every matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}_5)$ gives rise to a fractional linear transformation f_A defined by $f_A(x) = \frac{ax+b}{cx+d}$, and every fractional linear transformation f is expressible as f_A for some matrix $A \in GL_2(\mathbb{F}_5)$. However, more than one matrix A will in general express the same fractional linear transformation. For example, the matrix $A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ yields the fractional linear transformation $f_A(x) = \frac{2x+3}{3x+4}$; but the matrix $2A = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}$ yields the same fractional linear transformation $f_{2A} = f_A$ due to cancellation of 2's in the numerator and denominator. In fact $f_A = f_B$ iff $B = \lambda A$ for some nonzero λ, i.e. $\lambda \in \{1,2,3,4\}$.

   a. If $A, B \in GL_2(\mathbb{F}_5)$, show that $f_A \circ f_B = f_{AB}$. In particular, *the composite of any two fractional linear transformations is again a fractional linear transformation*.

In light of (a), the fractional linear transformations on X form a group. We denote this group $PGL_2(\mathbb{F}_5)$. By the remarks above, the map $GL_2(\mathbb{F}_5) \to PGL_2(\mathbb{F}_5)$ is a 4-to-1 surjection (it is 'onto' and 4-to-1).

   b. Determine $|PGL_2(\mathbb{F}_5)|$, the number of fractional linear transformations.
   c. Find explicit matrices $A, B \in GL_2(\mathbb{F}_5)$ such that $f_A = \alpha$ and $f_B = \beta$ where α and β are the particular permutations of X given in #2.
   d. Deduce that the group of legal moves of the puzzle in #2 is isomorphic to $PGL_2(\mathbb{F}_5)$.

      Hint: $PGL_2(\mathbb{F}_5)$ has a subgroup $\langle f_A , f_B \rangle = \langle \alpha, \beta \rangle$. Compare the order of this subgroup with the order of $PGL_2(\mathbb{F}_5)$.