Final Examination

December, 2023

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an 8.5″×11″ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Time permitted: 120 minutes. Total value of questions: 100 points (plus 20 bonus points).

1. *(15 points)* Let $G = S_6$, and consider the element $\sigma = (1,2,3)(4,5,6) \in G$.

   (a) How many conjugates does σ have in G?

   (b) Describe its centralizer $H = C_G(\sigma)$. What is its order $|H|$? Is H abelian? Is it cyclic?

   (c) Find an involution (element of order two) $\tau \in G$ such that $\langle \sigma, \tau \rangle$ is cyclic of order 6.

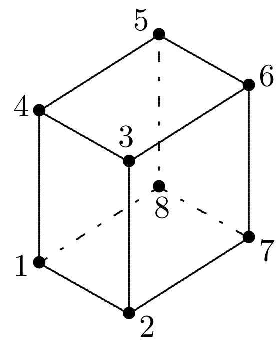
2. *(20 points)* Let $G = GL_n(\mathbb{R})$ where $n \geqslant 2$. For $g \in G$, denote the transpose of g by g^T. (Recall that g^T is the matrix with (i, j)-entry equal to the (j, i)-entry of g; also recall that $(gh)^T = h^T g^T$.)

   (a) Consider the map $\phi : G \to G$ defined by $\phi(g) = (g^T)^{-1}$. Show that ϕ is an automorphism of G.

   (b) Is the automorphism in (a) inner? That is, does there exist $w \in G$ such that $\phi(g) = wgw^{-1}$ for all $g \in G$? Explain.

(c) Now consider the subgroup $H = SL_2(\mathbb{R}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \;:\; a, b, c, d \in \mathbb{R},\ ad - bc = 1 \right\} < GL_2(\mathbb{R})$. Show that the map $\phi(g) = (g^T)^{-1}$ defines an inner automorphism of H.

3. *(20 points)* Consider the cube with vertices labelled $1, 2, \ldots, 8$ as shown (the vertex 8 is on the far side of the cube). Let G be the symmetry group of the cube, represented as a subgroup of S_8 using its action on the eight vertices. Recall that $|G| = 48$.



(a) Find an element $\sigma \in G$ such that $\sigma(1) = 4$, $\sigma(2) = 5$ and $\sigma(3) = 6$. (Express $\sigma \in S_8$ using the standard cycle notation for permutations.) What is the order of σ?

(b) Show that σ is conjugate to its inverse in G.

(c) Does σ preserve or reverse orientation? (Equivalently, is σ a rotation or not?)

(d) Determine the centralizer $C_G(\sigma)$, indicating in particular the order and the structure of this subgroup.

4. *(15 points)* There exists a homomorphism $\phi : S_4 \to S_4$ satisfying $\phi\big((1,2,3)\big) = (1,3,2)$ and $\phi\big((3,4)\big) = (2,3)$ (you may assume this fact as given).

(a) Is ϕ surjective? What is the image of ϕ?

(b) Determine $\ker\phi = \{\sigma \in S_4 : \phi(\sigma) = ()\}$. What is the order of this kernel?

(c) Is there a *unique* homomorphism $\phi : S_4 \to S_4$ having the two values specified? Explain.

5. *(20 points)* Let $G = GL_2(\mathbb{F}_5)$, the group of all invertible 2×2 matrices with entries in the field of order 5, $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ (the integers mod 5).

(a) What is the order $|G|$ of this group?

(b) Find an element of order 5 in G.

(c) Find an element of order 3 in G.

(d) Find an element of order 6 in G.

6. *(30 points)* Answer TRUE or FALSE to each of the following statements.

(a) Every finite group is isomorphic to a subgroup of $GL_n(\mathbb{R})$ for some $n \geq 1$.

______*(True/False)*

(b) Every finite group G has the form $G = \langle g, h \rangle$ for two elements $g, h \in G$.

______*(True/False)*

(c) If G and H are groups, then every subgroup of the direct product $G \times H$ has the form $A \times B$ for some subgroups $A \leqslant G$ and $B \leqslant H$.

______*(True/False)*

(d) If n is any positive integer, then there exists a group G containing elements g, h of order 2 such that gh has order n. ______*(True/False)*

(e) If G is an abelian group, then the set of all elements $g \in G$ of finite order is necessarily a subgroup of G. ______*(True/False)*

(f) If G is a nonabelian group of infinite order, then there exists at least one element in G of infinite order. ______*(True/False)*

(g) The alternating group A_5 has a normal subgroup of order 4. ______*(True/False)*

(h) If p is prime, then any two groups of order p are necessarily isomorphic.

______*(True/False)*

(i) For every positive integer n, the group $G = SL_2(\mathbb{R})$ has an element of order n.

______*(True/False)*

(j) Given a left coset $gH \subseteq G$ of a subgroup $H \leqslant G$, if gH is also a left coset of a subgroup $K \leqslant G$, then necessarily $K = H$. ______*(True/False)*