Analysis I (Math 3205)
Fall 2020

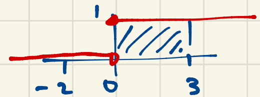
Book 2

$$\left\{\begin{matrix}\text{Lebesgue}\\ \text{integrable}\\ \text{functions}\end{matrix}\right\} \supset \left\{\begin{matrix}\text{Riemann}\\ \text{integrable}\\ \text{functions}\end{matrix}\right\} \supset \left\{\begin{matrix}\text{continuous}\\ \text{functions}\end{matrix}\right\} \supset \left\{\begin{matrix}\text{differentiable}\\ \text{functions}\end{matrix}\right\} \supset \left\{\begin{matrix}\text{continuously}\\ \text{differentiable}\\ \text{functions}\end{matrix}\right\} \supset \cdots$$

$\Downarrow$ g Dirichlet function

$\Downarrow$ H Heaviside function

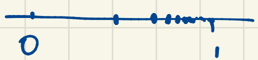
$$H(x) = \begin{cases} 1 & \text{if } x \geq 0; \\ 0 & \text{if } x < 0 \end{cases}$$



$$\int_{-2}^{3} H(x)\,dx = 3$$

$S = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots\}$ is bounded: 0 is a lower bound, 1 is an upper bound.

$\frac{1}{2}$ is the greatest lower bound.



Every $m \leq \frac{1}{2}$ is a lower bound for S, meaning $s \geq \frac{1}{2}$ for all $s \in S$.

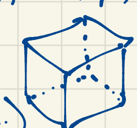
(so $\frac{1}{2}$ is a lower bound) and it is the greatest lower bound.

1 is the least upper bound of S.

Fact: $|[0,1]| = |\mathbb{R}^3|$

Basic idea of the ~~proof~~: $|(0,1)| = |(0,1)^3|$

$(0,1)^3 = (0,1) \times (0,1) \times (0,1) = \{(x,y,z) : 0 < x,y,z < 1\}$



Bijection: $a \longmapsto (0.a_1a_4a_7a_{10}a_{13}\cdots,\ 0.a_2a_5a_8a_{11}a_{14}\cdots,\ 0.a_3a_6a_9a_{12}a_{15}\cdots)$

$0 < a < 1$

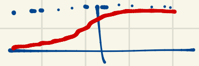
$0.14159265358\ldots \mapsto (0.1565\ldots,\ 0.4958\ldots,\ 0.1239\ldots)$

$\pi - 3$

$a = 0.a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}\cdots$

$= a_1 \cdot 10^{-1} + a_2 \cdot 10^{-2} + a_3 \cdot 10^{-3} + a_4 \cdot 10^{-4} + \cdots, \quad a_i \in \{0,1,2,\dots,9\}$

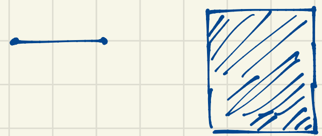
$|(0,1)| = |\mathbb{R}|$



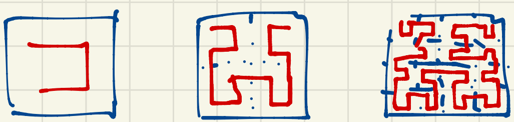
see video on Cardinality

$|(0,1)| = |(0,1)^3| = |\mathbb{R}^3|$ and this bijection can be given constructively i.e. by an explicit formula (in particular this is a theorem in ZF, not requiring the Axiom of Choice)

$|[0,1]| = |[0,1]^2|$

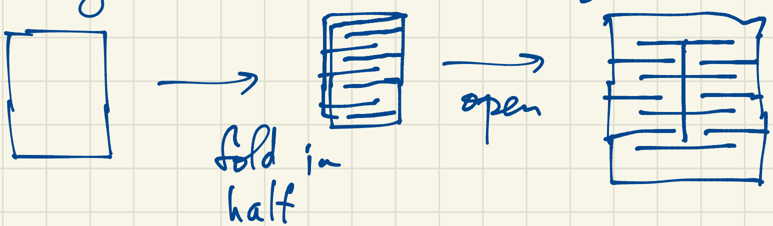


There is a bijection $[0,1] \to [0,1]^2$ but no continuous bijection.
However there is a continuous surjection (map that is onto)
This gives a "space-filling curve": it goes through every point of the square.



etc.

How do you cut a hole in an $8\frac{1}{2}'' \times 11''$ sheet of paper that you can walk through?



Fact: There is a set of open intervals in $\mathbb{R}$ of total length less than 1 which covers all the rational numbers.

Since $\mathbb{Q}$ is countable, $\mathbb{Q} = \{a_1, a_2, a_3, a_4, a_5, \dots\}$. Then

$$\mathbb{Q} \subseteq \bigcup_{n=1}^{\infty} \left(a_n - \frac{1}{2^{n+1}}, a_n + \frac{1}{2^{n+1}}\right) = \left(a_1 - \tfrac{1}{4}, a_1 + \tfrac{1}{4}\right) \cup \left(a_2 - \tfrac{1}{8}, a_2 + \tfrac{1}{8}\right) \cup \left(a_3 - \tfrac{1}{16}, a_3 + \tfrac{1}{16}\right) \cup \left(a_4 - \tfrac{1}{32}, a_4 + \tfrac{1}{32}\right) \cup \cdots$$

$$\text{Total length} < \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots = 1$$

Once again the set of intervals can be given constructively ie. explicitly with no need for the Axiom of Choice.

What is a (Riemann) integral? i.e. the integral as defined in Calculus I-II?

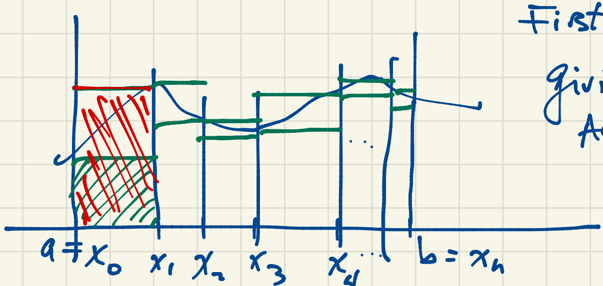
Suppose $f: [a,b] \to \mathbb{R}$. We want to define $\int_a^b f(x)\,dx$. We start with lower and upper bounds for the integral (these being upper and lower Riemann sums).

We then take $\sup\{\text{lower Riemann sums}\}$ and $\inf\{\text{upper Riemann sums}\}$.

First subdivide $[a,b]$ at points $a = x_0 \le x_1 \le x_2 \le \cdots \le x_n = b$ giving n subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \dots, n$.

Assuming f is bounded, $m_i \le f(x) \le M_i$ on $[x_{i-1}, x_i]$

where $M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\} = \sup_{[x_{i-1}, x_i]} f$, $m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\} = \inf_{[x_{i-1}, x_i]} f$



The Riemann Sums corresponding to the partition $a = x_0 \le x_1 \le x_2 \le \cdots \le x_{n-1} \le x_n = b$ of $[a,b]$ are:

Upper sum $\sum_{i=1}^{n} \underbrace{(x_i - x_{i-1})}_{\text{base}} \underbrace{M_i}_{\text{height}} = (x_1 - x_0)M_1 + (x_2 - x_1)M_2 + \cdots + (x_n - x_{n-1})M_n$.

Lower sum $\sum_{i=1}^{n} (x_i - x_{i-1}) m_i$

We should have $\underset{\sum (x_i - x_{i-1})m_i}{\text{Lower sum}} \le \int_a^b f(x)\,dx \le \underset{\sum (x_i - x_{i-1})M_i}{\text{Upper sum}}$

We can't just let $n \to \infty$. By the Least Upper Bound Property, sup{lower bounds} exists and inf{upper bounds} exists. And

$$\sup\{\text{lower bounds}\} \le \inf\{\text{upper bounds}\}.$$

If these two values agree, this gives a definite value for $\int_a^b f(x)\,dx$.

For lots of functions (eg. the Heaviside function and for all continuous functions), this works. For Dirichlet's function, the Riemann integral $\int_0^1 g(x)\,dx$ is undefined.

Why? For Dirichlet's function $g(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$

$m_i = 0, \quad M_i = 1 \quad$ for each $i = 1, 2, \dots, n$

Upper sums: $\displaystyle\sum_{i=1}^{n} (x_i - x_{i-1}) \underbrace{M_i}_{1} = (\cancel{x_1} - x_0) + (\cancel{x_2} - \cancel{x_1}) + (\cancel{x_3} - \cancel{x_2}) + \cdots + (\cancel{x_{n-1}} - \cancel{x_{n-2}}) + (x_n - \cancel{x_{n-1}})$

$= x_n - x_0 = b - a = 1 - 0 = 1$

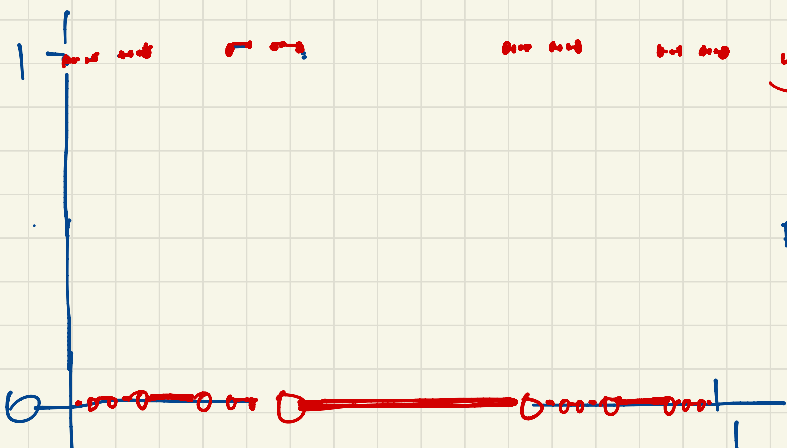
Lower sums: $\displaystyle\sum_{i=1}^{m} (x_i - x_{i-1}) \underbrace{m_i}_{0} = 0 + 0 + \cdots + 0 = 0.$

For C the Cantor set define $u(x) = \begin{cases} 1, & \text{if } x \in C \\ 0, & \text{if } x \notin C. \end{cases}$

What is $\displaystyle\int_0^1 u(x)\,dx$?

Lower sums are 0.

Upper sums can be made as small as we want.



$y = u(x)$

For $\underbrace{0 < \tfrac{1}{3}}_{1} \underbrace{< \tfrac{2}{3}}_{0} \underbrace{< 1}_{1}$ the Riemann Sum is $\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = \frac{2}{3}$.

For $0 < \frac{1}{9} < \frac{2}{9} < \frac{1}{3} < \frac{2}{3} < \frac{7}{9} < \frac{8}{9} < 1$ the Riemann Sum is

$$\frac{1}{9} \cdot 1 + \frac{1}{9} \cdot 0 + \frac{1}{9} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{9} \cdot 1 + \frac{1}{9} \cdot 0 + \frac{1}{9} \cdot 1 = \frac{4}{9}$$

For $0 < \frac{1}{27} < \frac{2}{27} < \cdots < \frac{26}{27} < 1$ the corresponding Riemann Sum is $\frac{8}{27}$.

Each new upper Riemann Sum is $\frac{2}{3}$ of the previous one i.e. $\left(\frac{2}{3}\right)^n$.

$\inf\{\text{upper Sums}\} = 0.$

For the function $u(x)$, $\underbrace{\sup\{\underbrace{\text{lower sums}}_{0}\}}_{0} \leq \int_0^1 u(x)\,dx \leq \underbrace{\inf\{\text{upper sums}\}}_{0}$

So $\int_0^1 u(x)\,dx = 0$.

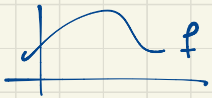
Note: $u(x)$ has infinitely many discontinuities but it is not discontinuous everywhere.
$u(x)$ is continuous on a set of open intervals inside $[0,1]$ of total length 1.
The total length of the Cantor set C (where $u=1$) is 0.
However C is uncountable; $|C| = |\mathbb{R}|$. Why?
Every $a \in [0,1]$ has a ternary expansion

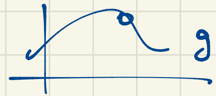
$$a = 0.a_1a_2a_3a_4a_5\cdots \qquad (a_i \in \{0,1,2\})$$
$$= \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \frac{a_4}{3^4} + \frac{a_5}{3^5} + \cdots$$

The points in C are those with $a_i \in \{0,2\}$ only.
$|C| = |\mathbb{R}| = |[0,1]|$. A bijection $C \longrightarrow [0,1]$

$$\underset{\substack{\text{(base 3)}\\ \text{ternary}}}{0.20022202002\cdots} \longmapsto \underset{\substack{\text{(base 2)}\\ \text{binary}}}{0.10011101001\cdots}$$

If two functions f and $\dot{g}$ agree except at a single point, the $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$





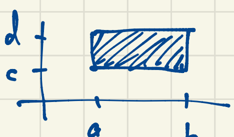
The same holds for changing a function at any finite number of points.

We want to be able to <u>measure</u> sets to distinguish their size, not as cardinality, but length (in one dimension), area (in two dimensions), volume (in 3 dimensions), etc.
Defining measure of a set is equivalent to being able to integrate.
In one dimension, $\lambda([a,b]) = b-a$ for $a \le b$. (the length)

Greek letter lambda)

In two dimensions, $\lambda([a,b] \times [c,d]) = (b-a)(d-c)$

Cartesian product $\{(x,y) : x \in [a,b],\ y \in [c,d]\}$.



Borel measure extends this notion to larger sets and more complicated constructions.
Borel measure extends to Lebesgue measure which is the gold standard for measuring sets.

Lebesgue measure of $A \subseteq \mathbb{R}^n$ is denoted $\lambda(A)$.

$\lambda([a,b]) = b-a$ for $a \le b$

$\lambda(A) \geq 0$ for all measurable A.

$\lambda(\{a\}) = 0$

$\lambda(A \sqcup B) = \lambda(A) + \lambda(B)$

$\sqcup$ disjoint union.

This extends to countable disjoint unions:

$$\lambda\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \lambda(A_i)$$

If $A \subseteq B$ then $\lambda(A) \leq \lambda(B)$.

$\lambda(\mathbb{Q}) = 0$. This follows from the properties above:

$$\mathbb{Q} = \{a_1, a_2, a_3, \dots\} = \bigsqcup_{i=1}^{\infty} \{a_i\}$$

$\{a_i\}$ ← singleton sets (sets with single elements)

$$\Rightarrow \lambda(\mathbb{Q}) = \sum_{i=1}^{\infty} \lambda(\{a_i\})$$

$$= \sum_{i=1}^{\infty} 0 = 0.$$

$\mathbb{R} = \bigsqcup_{a \in \mathbb{R}} \{a\}$ but this is not a countable union so $\lambda(\mathbb{R}) \neq 0$.

Recall, as observed about 5 slides back,

$$\mathbb{Q} \subset \underbrace{\bigcup_i \left(a_i - \frac{1}{2^{i+1}}, a_i + \frac{1}{2^{i+1}}\right)}_{\text{set of Lebesgue measure } < 1.}$$

<u>Sets of measure zero</u> are sets which can be covered by countable unions of intervals of total length as small as we want (ie. for every $\varepsilon > 0$, the set is covered by intervals of total length $< \varepsilon$). Such sets are considered 'negligible' in the sense of length ie. measure.

Sets of Lebesgue measure zero have $\lambda(A) = 0$.

eg. $\lambda(\mathbb{Q}) = 0$ so $\mathbb{Q}$ has Lebesgue measure zero.

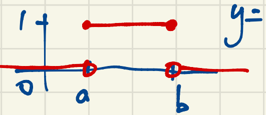
Also the Cantor set $C \subset [0,1]$ has measure zero.

$\mathbb{Q}$ is countable and C is uncountable so from the perspective of cardinality, there is a big difference in size between these two sets. But in terms of length (Lebesgue measure), both have measure zero: $\lambda(\mathbb{Q}) = \lambda(C) = 0$.

Connection between measure and integration:

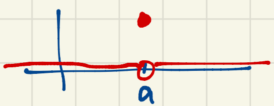
Given a set $A \subseteq \mathbb{R}$, its <u>characteristic</u> <u>function</u> $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

Greek "chi"



$y = \chi_{[a,b]}(x)$

$$\int_{-\infty}^{\infty} \chi_{[a,b]}(x)\,dx = b-a = \lambda([a,b])$$



$$\int_{-\infty}^{\infty} \chi_{\{a\}}(x)\,dx = 0 = \lambda(\{a\})$$

Similarly,

$$\int_{-\infty}^{\infty} \chi_C(x)\,dx = \lambda(C) = 0$$

where $C \subset [0,1]$ is the Cantor set

(and this integral is defined as both a Riemann integral and as a Lebesgue integral).

In general, $\displaystyle\int_{-\infty}^{\infty} \chi_A(x)\,dx = \lambda(A)$.

$\chi_{\mathbb{Q}} = g$ is Dirichlet's function

$$\int_{-\infty}^{\infty} \chi_{\mathbb{Q}}(x)\,dx = \lambda(\mathbb{Q}) = 0.$$

(This however is the <u>Lebesgue</u> <u>integral</u>, not the Riemann integral of Calculus I and II). The Riemann integral is undefined.

If f and g agree except at a finite number of points, $\underbrace{\int_a^b f(x)\,dx}_{\int_{[a,b]} f} = \int_a^b g(x)\,dx$

More generally, if f and g agree almost everywhere (i.e. except on a set of measure zero) then $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ for every interval $[a,b]$.

f and g agree almost everywhere (f and g agree a.e.)

$$\iff \lambda(\{x \in \mathbb{R} : f(x) \neq g(x)\}) = 0$$

This is an important example of an equivalence relation.

If $f = g$ a.e. and $g = h$ a.e. then $f = h$ a.e.

$f = f$ a.e.

If $f = g$ a.e. then $g = f$ a.e.

$\lambda(A \sqcup B) = \lambda(A) + \lambda(B)$ for all measurable sets A, B.

If B is a closed unit ball in $\mathbb{R}^3$ then $\lambda(B) = \frac{4\pi}{3}$ (volume)

radius 1

$B = A_1 \sqcup \cdots \sqcup A_5$ where $A_1, \cdots, A_5$ can be repositioned to form two unit balls of total Lebesgue measure (volume) $\frac{8\pi}{3}$.

$\lambda(B) \stackrel{?}{=} \underbrace{\cancel{\lambda(A_1) + \cdots + \lambda(A_5)}}_{\text{undefined}} = 2\lambda(B)$.

$A_1, \cdots, A_5$ are non-measurable.