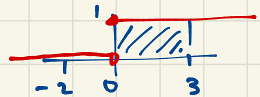
Analysis I (Math 3205)
Fall 2020

Book 2

$\left\{\begin{matrix}\text{Lebesgue}\\ \text{integrable}\\ \text{functions}\end{matrix}\right\} \supset \left\{\begin{matrix}\text{Riemann}\\ \text{integrable}\\ \text{functions}\end{matrix}\right\} \supset \left\{\begin{matrix}\text{continuous}\\ \text{functions}\end{matrix}\right\} \supset \left\{\begin{matrix}\text{differentiable}\\ \text{functions}\end{matrix}\right\} \supset \left\{\begin{matrix}\text{continuously}\\ \text{differentiable}\\ \text{functions}\end{matrix}\right\} \supset \cdots$

$\Downarrow$ g Dirichlet function $\qquad \Downarrow$ H Heaviside function

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0; \\ 0 & \text{if } x < 0 \end{cases}$$



$$\int_{-2}^{3} H(x)\,dx = 3$$

$S = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots\}$ is bounded: 0 is a lower bound, 1 is an upper bound.

$\frac{1}{2}$ is the greatest lower bound.

Every $m \leq \frac{1}{2}$ is a lower bound for S, meaning $s \geq \frac{1}{2}$ for all $s \in S$.

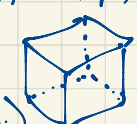
(so $\frac{1}{2}$ is a lower bound) and it is the greatest lower bound.

1 is the least upper bound of S.

Fact: $|[0,1]| = |\mathbb{R}^3|$

Basic idea of the proof: $|(0,1)| = |(0,1)^3|$

$(0,1)^3 = (0,1) \times (0,1) \times (0,1) = \{(x,y,z) : 0 < x,y,z < 1\}$



Bijection: $a \longmapsto (0.a_1 a_4 a_7 a_{10} a_{13} \cdots,\ 0.a_2 a_5 a_8 a_{11} a_{14} \cdots,\ 0.a_3 a_6 a_9 a_{12} a_{15} \cdots)$

$0 < a < 1$

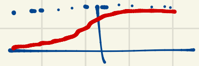
$0.14159265358\ldots \longmapsto (0.1565\ldots,\ 0.4958\ldots,\ 0.1239\ldots)$

$\pi - 3$

$a = 0.a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} \cdots$

$= a_1 \cdot 10^{-1} + a_2 \cdot 10^{-2} + a_3 \cdot 10^{-3} + a_4 \cdot 10^{-4} + \cdots, \quad a_i \in \{0,1,2,\dots,9\}$

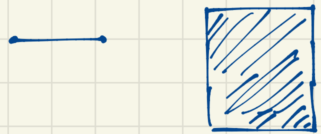
$|(0,1)| = |\mathbb{R}|$



see video on Cardinality

$|(0,1)| = |(0,1)^3| = |\mathbb{R}^3|$ and this bijection can be given constructively i.e. by an explicit formula (in particular this is a theorem in ZF, not requiring the Axiom of Choice)

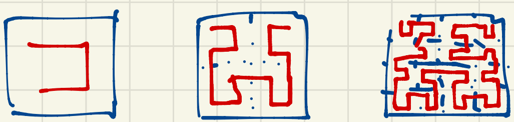
$|[0,1]| = |[0,1]^2|$



There is a bijection $[0,1] \to [0,1]^2$ but no continuous bijection.

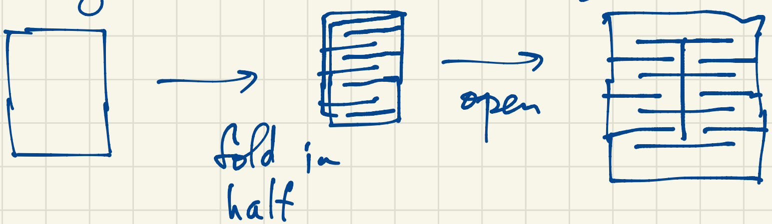
However there is a continuous surjection (map that is onto)

This gives a "space-filling curve": it goes through every point of the square.



etc.

How do you cut a hole in an $8\frac{1}{2}'' \times 11''$ sheet of paper that you can walk through?



Fact: There is a set of open intervals in $\mathbb{R}$ of total length less than 1 which covers all the rational numbers.

Since $\mathbb{Q}$ is countable, $\mathbb{Q} = \{a_1, a_2, a_3, a_4, a_5, \dots\}$. Then

$$\mathbb{Q} \subseteq \bigcup_{n=1}^{\infty} \left(a_n - \tfrac{1}{2^{n+1}}, a_n + \tfrac{1}{2^{n+1}}\right) = \left(a_1 - \tfrac{1}{4}, a_1 + \tfrac{1}{4}\right) \cup \left(a_2 - \tfrac{1}{8}, a_2 + \tfrac{1}{8}\right) \cup \left(a_3 - \tfrac{1}{16}, a_3 + \tfrac{1}{16}\right) \cup \left(a_4 - \tfrac{1}{32}, a_4 + \tfrac{1}{32}\right) \cup \dots$$

Total length $< \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1$

Once again the set of intervals can be given constructively i.e. explicitly with no need for the Axiom of Choice.

What is a (Riemann) integral? i.e. the integral as defined in Calculus I-II?

Suppose $f: [a,b] \to \mathbb{R}$. We want to define $\int_a^b f(x)\,dx$. We start with lower and upper bounds for the integral (these being upper and lower Riemann sums).

We then take $\sup\{\text{lower Riemann sums}\}$ and $\inf\{\text{upper Riemann sums}\}$.