

Analysis I (Math 3205)

Fall 2020

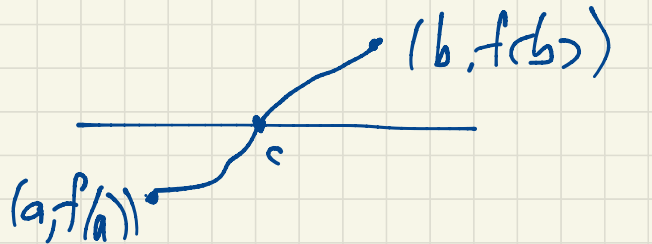
Book I

Intermediate Value Theorem

If $f: [a, b] \rightarrow \mathbb{R}$ is continuous with $f(a) < 0 < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = 0$.

How does anyone prove this?

What is your experience with reading/writing proofs?

 The theorem does not hold over \mathbb{Q} .

eg. $f(x) = x^2 - 2$, $f: \{ \text{rationals between } 0 \text{ and } 2 \} \rightarrow \mathbb{Q}$ is continuous, $f(0) < 0 < f(2)$ but there is no solution of $f(c) = 0$ in \mathbb{Q} .

\mathbb{Q} is not complete; \mathbb{R} is complete.

The complete statement of the Intermediate Value Theorem: For all $f: [a, b] \rightarrow \mathbb{R}$, if f is continuous and $f(a) < 0 < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = 0$.

"for all", "for every", "for each": universal quantifiers

"there is", "there exists": existential quantifiers.

For all x there exists y such that $x < y$. (True in \mathbb{R})

There exists y such that for all x , $x < y$.

Definition of a Limit

We say $\lim_{x \rightarrow a} f(x) = L$ if the following condition holds:

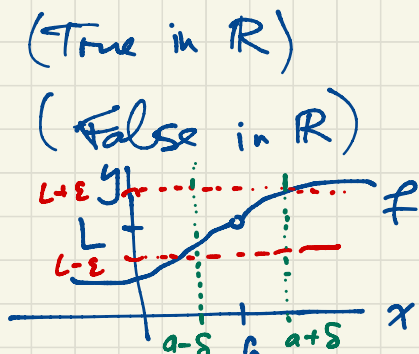
For all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{|f(x) - L| < \varepsilon}_{f(x) \text{ is within } \varepsilon \text{ of } L} \text{ whenever } \underbrace{0 < |x - a| < \delta}_{x \text{ is within } \delta \text{ of } a}.$$

$f(x)$ is within ε of L

x is within δ
of a

ie. for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all x , if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.



Note: It doesn't matter here what $f(a)$ is or even whether or not it's defined

Let's prove that $\lim_{x \rightarrow 2} (5x+1) = 11$.

Rough version: $f(x) = 5x+1$. If we need $f(x)$ to be within ε of 11, how close does x have to be to 2?

$$|f(x) - 11| < \varepsilon \iff 11 - \varepsilon < f(x) < 11 + \varepsilon$$

$$\iff 11 - \varepsilon < 5x+1 < 11 + \varepsilon$$

$$\iff 10 - \varepsilon < 5x < 10 + \varepsilon$$

$$\iff 2 - \frac{\varepsilon}{5} < x < 2 + \frac{\varepsilon}{5}$$

$$\iff |x-2| < \frac{\varepsilon}{5}$$

Proof

(Actually): Let $\varepsilon > 0$. Then whenever $0 < |x-2| < \frac{\varepsilon}{5}$ we have $2 - \frac{\varepsilon}{5} < x < 2 + \frac{\varepsilon}{5}$ so

$$11 - \varepsilon < 5x+1 < 11 + \varepsilon \quad \text{i.e.} \quad |f(x) - 11| < \varepsilon.$$

Another proof: Suppose $\lim_{x \rightarrow 7} f(x) = 4$ and $\lim_{x \rightarrow 7} g(x) = 5$. Prove that $\lim_{x \rightarrow 7} (f(x) + g(x)) = 9$.

Rough version: Given $\varepsilon > 0$ we must find $\delta > 0$ such that $|f(x) + g(x) - 9| < \varepsilon$ whenever $0 < |x-7| < \delta$. Since $\lim_{x \rightarrow 7} f(x) = 4$, we can find $\delta > 0$ such that $|f(x) - 4| < \varepsilon$

whenever $0 < |x-7| < \delta$. Also since $\lim_{x \rightarrow 7} g(x) = 5$, we can find δ' such that

$$|g(x) - 5| < \varepsilon \quad \text{whenever} \quad 0 < |x-7| < \delta'$$

$$4 - \varepsilon < f(x) < 4 + \varepsilon \quad \text{whenever} \quad 7 - \delta < x < 7 + \delta$$

$$4 - \varepsilon < f(x) < 4 + \varepsilon \quad \text{whenever} \quad 7 - \delta < x < 7 + \delta \quad \text{i.e. } |x - 7| < \delta$$

$$5 - \varepsilon < g(x) < 5 + \varepsilon \quad \text{whenever} \quad 7 - \delta' < x < 7 + \delta' \quad \text{i.e. } |x - 7| < \delta'$$

$$\underbrace{4 - 2\varepsilon < f(x) + g(x) < 4 + 2\varepsilon}_{\text{whenever } 0 < |x - 7| < \min\{\delta, \delta'\}}$$

$$|f(x) + g(x) - 9| < 2\varepsilon \quad \text{whenever} \quad 0 < |x - 7| < \min\{\delta, \delta'\}.$$

Actual (final) proof:

Let $\varepsilon > 0$. There exists δ such that $|f(x) - 4| < \frac{\varepsilon}{2}$ whenever $0 < |x - 7| < \delta$. Also there exists $\delta' > 0$ such that $|g(x) - 5| < \frac{\varepsilon}{2}$ whenever $0 < |x - 7| < \delta'$. Then

$$|f(x) + g(x) - 9| \leq |f(x) - 4| + |g(x) - 5| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $0 < |x - 7| < \min\{\delta, \delta'\}$. □

Note: The triangle inequality says $|a + b| \leq |a| + |b|$ for all a, b .

$$|f(x) + g(x) - 9| = |f(x) - 4 + g(x) - 5|$$

$$|f(x) - 4 + g(x) - 5| \leq |f(x) - 4| + |g(x) - 5|$$

$$|f(x) + g(x) - 9| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \curvearrowright$$

$$|f(x) + g(x) - 9| < \varepsilon$$

$$7-2=5 \iff 7-2=\boxed{5}$$
$$\iff 7=\boxed{5}+2$$

$$\infty+1=\infty \iff \infty-\infty=1$$

$$\infty+2=\infty \iff \infty-\infty=2$$

$$\} \Rightarrow 1=2.$$

$\lim_{x \rightarrow \infty} f(x) = \infty$ means for all M there exist N such that $f(x) > M$ whenever $x > N$.

$$[N, \infty) = \{a \in \mathbb{R} : a \geq N\}$$

$$(N, \infty) = \{a \in \mathbb{R} : a > N\}$$

$$0.9999999\dots = 1$$

$$\frac{1}{3} = 0.333333\dots$$

$$\text{Eg. } g(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

For $x \neq 0$, $-1 \leq \sin \frac{1}{x} \leq 1$
 so $-x^2 \leq \underbrace{x^2 \sin \frac{1}{x}}_{g(x)} \leq x^2$

Why is g continuous at 0?

$$\lim_{x \rightarrow 0} g(x) = 0 \text{ because } -x^2 \leq g(x) \leq x^2$$

where $\lim_{x \rightarrow 0} (-x^2) = 0 = \lim_{x \rightarrow 0} x^2$ so we can use the Squeeze Theorem.

Since $\lim_{x \rightarrow 0} g(x) = 0 = g(0)$, g is continuous at 0.

Is g differentiable at 0?

$$\begin{aligned} g'(0) &= \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 \sin \frac{1}{h} - 0}{h} \\ &= \lim_{h \rightarrow 0} h \sin \frac{1}{h} = 0 \end{aligned}$$

$\lim_{h \rightarrow 0} \frac{g(a+h) - g(a-h)}{2h}$? slope of secant
 line between $(a-h, g(a-h))$ and $(a+h, g(a+h))$



$$g'(a) = \lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h}$$

by the Squeeze Theorem since $-|h| \leq h \sin \frac{1}{h} \leq |h|$ whenever $h \neq 0$ $-1 \leq \sin \frac{1}{h} \leq 1$
 where $\lim_{h \rightarrow 0} |h| = 0 = \lim_{h \rightarrow 0} (-|h|)$. $-|h| \leq h \sin \frac{1}{h} \leq |h|$

$$g(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

$$g'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Uses the chain rule and the product rule

If $x \neq 0$, $g'(x) = x^2 \left(\cos \frac{1}{x} \right) \left(-\frac{1}{x^2} \right) + 2x \sin \frac{1}{x} = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$.

Note: g is differentiable (i.e. everywhere in its domain which is \mathbb{R}).

It is not possible to evaluate $g'(0)$ using the chain rule, product rule, rules for derivatives of power functions and trig functions, etc.

$\lim_{x \rightarrow 0} g'(x)$ does not exist. So g' is not continuous at 0.

g is differentiable but not continuously differentiable.

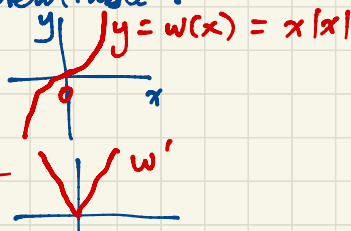
Example of a function which is differentiable but not twice differentiable:

$g(x)$ as above; also $w(x) = x|x| = \begin{cases} x^2, & \text{if } x \geq 0; \\ -x^2, & \text{if } x < 0. \end{cases}$

(check: $w'(x) = 2|x|$)

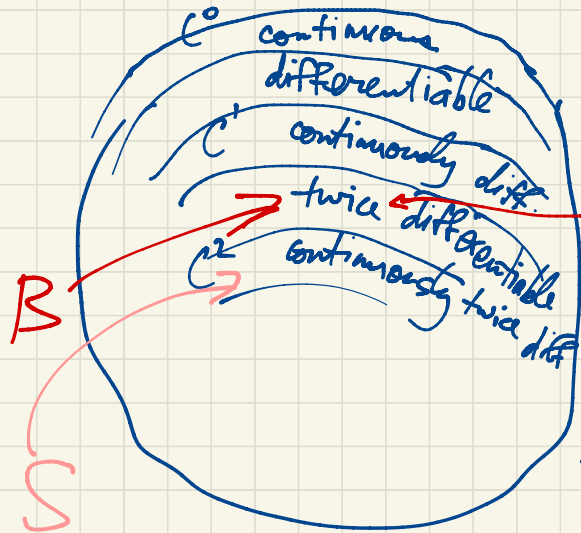
$w''(0)$ does not exist.

w is continuously differentiable but not twice differentiable.



Hierarchy of Smoothness:

$C^n = C^n(\mathbb{R})$ is the set of functions $\mathbb{R} \rightarrow \mathbb{R}$ which are continuously n times differentiable.



What is an example of a function that is twice differentiable but not continuously twice differentiable? Find $f: \mathbb{R} \rightarrow \mathbb{R}$ such that f'' is defined everywhere but f'' is not continuous.

$$f(x) = \begin{cases} x(\ln|x| - 1), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x(\ln|x| - 1) = 0$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} x(\ln x - 1) = \lim_{x \rightarrow 0^+} \frac{\ln x - 1}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\ &= \lim_{x \rightarrow 0^+} (-x) = 0 \end{aligned}$$

f is now continuous; is it differentiable?

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \frac{h(\ln|h| - 1) - 0}{h} = \lim_{h \rightarrow 0} (\ln|h| - 1) = -\infty$$

so this function is not differentiable.

$V(x) = \sin \frac{1}{x^2+1}$ is C^∞ i.e. $V \in C^\infty(\mathbb{R})$ (this function is infinitely differentiable on all of \mathbb{R})

$$V'(x) = \cos \frac{1}{x^2+1} \cdot \left(\frac{(x^2+1) \cdot 0 - 1 \cdot 2x}{(x^2+1)^2} \right) = \frac{-2x}{(1+x^2)^2} \cos \frac{1}{1+x^2}$$

If we continue taking higher and higher order derivatives, every $V^{(n)}(x)$ has terms of the form $\frac{\text{polynomial in } x}{(x^2+1)^k} \cdot \begin{matrix} \sin(\frac{1}{x^2+1}) \\ \uparrow \\ \text{or } \cos \end{matrix}$

$$R(x) = \begin{cases} x^3 \sin \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

$$\text{If } x \neq 0, R'(x) = 3x^2 \sin \frac{1}{x} + x^3 \left(\cos \frac{1}{x} \right) \left(-\frac{1}{x^2} \right) = 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}.$$

$$R'(0) = \lim_{h \rightarrow 0} \frac{R(h) - R(0)}{h} = \lim_{h \rightarrow 0} \frac{h^3 \sin \frac{1}{h} - 0}{h} = \lim_{h \rightarrow 0} h^2 \sin \frac{1}{h} = 0$$


$$R'(x) = \begin{cases} 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases} \quad (\text{see Tuesday's class})$$

$$R''(x) = 6x \sin \frac{1}{x} + 3x^2 \cos \frac{1}{x} \cdot \left(-\frac{1}{x^2}\right) - \cos \frac{1}{x} + x \sin \frac{1}{x} \cdot \left(-\frac{1}{x^2}\right)$$
$$= 6x \sin \frac{1}{x} - 4 \cos \frac{1}{x} - \frac{1}{x} \sin \frac{1}{x}, \quad \text{if } x \neq 0.$$

$$R''(0) = \lim_{h \rightarrow 0} \frac{R'(h) - R'(0)}{h} = \lim_{h \rightarrow 0} \frac{3h^2 \sin \frac{1}{h} - h \cos \frac{1}{h} - 0}{h}$$

$$= \lim_{h \rightarrow 0} \left(3h \sin \frac{1}{h} - \cos \frac{1}{h} \right) \text{ does not exist.}$$

So R is not twice differentiable.

Try $S(x) = x^2|x|$ 

$$S'(x) = \lim_{h \rightarrow 0} \frac{S(x+h) - S(x)}{h}$$

If $x > 0$, $S(x) = x^3$, $S'(x) = 3x^2$, $S''(x) = 6x$.

If $x < 0$, $S(x) = -x^3$, $S'(x) = -3x^2$, $S''(x) = -6x$.

$$S'(0) = \lim_{h \rightarrow 0} \frac{h^2|h| - 0}{h} = \lim_{h \rightarrow 0} h|h| = 0.$$

Note that $S'(x) = 3x|x|$ for all x . So S' is continuous i.e. S is continuously differentiable ($S \in C^1(\mathbb{R})$).

$$S''(0) = \lim_{h \rightarrow 0} \frac{S'(h) - S'(0)}{h} = \lim_{h \rightarrow 0} \frac{3h|h| - 0}{h} = \lim_{h \rightarrow 0} 3|h| = 0$$

Note: $S''(x) = 6|x|$ for all x

$$\text{Try } B(x) = \begin{cases} x^4 \sin \frac{1}{x} & , \text{ if } x \neq 0; \\ 0 & , \text{ if } x = 0 \end{cases}$$

$B'(x)$ is defined for all x but not continuous at 0. since $\lim_{x \rightarrow 0} B'(x)$ does not exist.

$$\text{For } x \neq 0, \quad B'(x) = 4x^3 \sin \frac{1}{x} + x^4 \left(\cos \frac{1}{x} \right) \left(-\frac{1}{x^2} \right) = 4x^3 \sin \frac{1}{x} - x^2 \cos \frac{1}{x}$$

$$B''(x) = 12x^2 \sin \frac{1}{x} + 4x^3 \cos \frac{1}{x} \left(-\frac{1}{x^2} \right) - 2x \cos \frac{1}{x} - x^2 \left(-\sin \frac{1}{x} \right) \left(-\frac{1}{x^2} \right)$$

$$B'(0) = \lim_{h \rightarrow 0} \frac{B(h) - B(0)}{h} = \lim_{h \rightarrow 0} \frac{h^4 \sin \frac{1}{h} - 0}{h} = \lim_{h \rightarrow 0} h^3 \sin \frac{1}{h} = 0$$

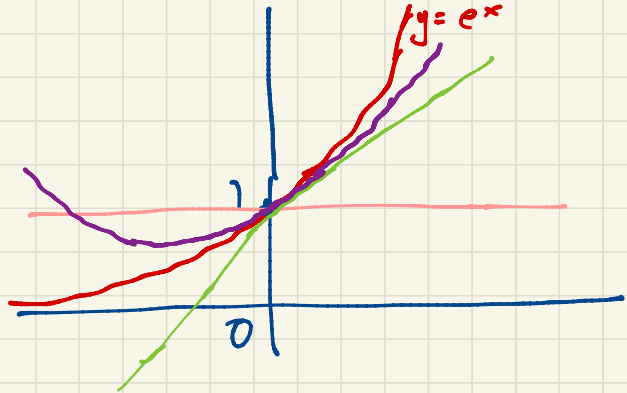
$= (12x^2 - 1) \sin \frac{1}{x} - 6x \cos \frac{1}{x}$

Note: $\lim_{x \rightarrow 0} B'(x) = \lim_{x \rightarrow 0} \left(4x^3 \sin \frac{1}{x} - x^2 \cos \frac{1}{x} \right) = 0 = B'(0)$ so B' is continuous

$$B''(0) = \lim_{h \rightarrow 0} \frac{B'(h) - B'(0)}{h} = \lim_{h \rightarrow 0} \frac{4h^3 \sin \frac{1}{h} - h^2 \cos \frac{1}{h} - 0}{h} = \lim_{h \rightarrow 0} \left(4h^2 \sin \frac{1}{h} - h \cos \frac{1}{h} \right) = 0$$

i.e. $B \in C^1(\mathbb{R})$ i.e. B is continuously differentiable

$$\lim_{x \rightarrow 0} B''(x) = \lim_{x \rightarrow 0} \left((12x^2 - 1) \sin \frac{1}{x} - 6x \cos \frac{1}{x} \right) \text{ does not exist!}$$



Order (0^{th} order approximation):

$$e^x \approx 1 \quad \text{for } x \approx 0.$$

Tangent line approximation: (1^{st} order approximation)

$$e^x \approx 1 + x \quad \text{for } x \approx 0.$$

Quadratic approximation (2^{nd} order approximation):

$$e^x \approx 1 + x + \frac{x^2}{2} \quad \text{for } x \approx 0$$

Taylor polynomial $T_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots + \frac{x^n}{n!}$

is the "best" polynomial approximation of degree n to the function e^x

For a general function $f \in C^n$, the Taylor polynomial of degree n is

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \dots + \frac{f^{(n)}(0)}{n!} x^n$$

$$n! = 1 \times 2 \times 3 \times \dots \times n.$$

$$\text{Eq. } f(x) = \frac{x}{1+2x}$$

$$f'(x) = \frac{(1+2x) \cdot 1 - x \cdot 2}{(1+2x)^2} = \frac{1}{(1+2x)^2}$$

$$f''(x) = \frac{(1+2x)^2 \cdot 0 - 1 \cdot 2(1+2x) \cdot 2}{(1+2x)^4} = \frac{-4}{(1+2x)^3}$$

$$f'''(x) = \frac{(1+2x)^3 \cdot 0 - (-4) \cdot 3(1+2x)^2 \cdot 2}{(1+2x)^6} = \frac{24}{(1+2x)^4}$$

$$f(0) = 0$$

$$f'(0) = 1$$

$$f''(0) = -4$$

$$f'''(0) = 24$$

etc.

The Taylor series for f centred at 0 is

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2} x^2 + \frac{f'''(0)}{6} x^3 + \dots$$

$$= 0 + x - \frac{4}{2} x^2 + \frac{24}{6} x^3 + \dots$$

$$= x - 2x^2 + 4x^3 - 8x^4 + 16x^5 - 32x^6 + \dots$$

Note: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$ (geometric series)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots \quad (\text{geometric series}) \quad \text{This converges for } |x| < 1.$$

$$\frac{1}{1+2x} = 1 - 2x + 4x^2 - 8x^3 + 16x^4 - 32x^5 + \dots \quad (\text{converges for } |-2x| < 1 \text{ i.e. } |x| < \frac{1}{2})$$

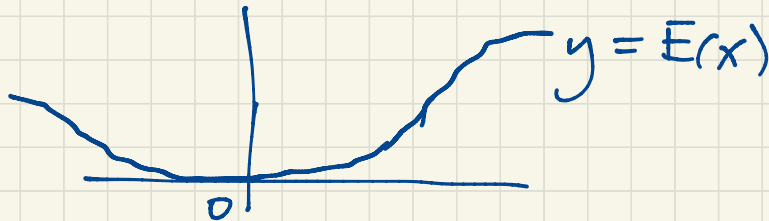
$$\frac{x}{1+2x} = x - 2x^2 + 4x^3 - 8x^4 + 16x^5 - 32x^6 + \dots \quad (\text{converges for } |x| < \frac{1}{2})$$

By the way, where does this series converge? (Review from Calc II)
Ratio Test.

Another example: $x^2 \sin x = x^2 \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \dots \right)$

$$= x^3 - \frac{x^5}{6} + \frac{x^7}{120} - \frac{x^9}{5040} + \dots \quad \text{for all } x.$$

$$\text{Eg. } E(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$



$$E(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases} \quad \frac{d}{dx} \left(-\frac{1}{x^2} \right) = \frac{d}{dx} \left(-x^{-2} \right) = 2x^{-3}$$

$$\text{If } x \neq 0, \quad E'(x) = e^{-1/x^2} (2x^{-3}) = 2x^{-3} e^{-1/x^2} = 2 \frac{e^{-1/x^2}}{x^3}$$

$$E''(x) = 2 \left(\frac{x^3 \cdot 2e^{-1/x^2} / x^3}{x^6} - e^{-1/x^2} \cdot 3x^2 \right)$$

$$= \frac{(4 - 6x^2) e^{-1/x^2}}{x^6}$$

$$E'''(x) = \frac{x^6 \cdot \left[(-12x) e^{-1/x^2} + (4 - 6x^2) \left(2e^{-1/x^2} / x^3 \right) - (4 - 6x^2) e^{-1/x^2} \cdot 6x^5 \right]}{x^{12}}$$

$$= \frac{x^6 (-12x) + (4 - 6x^2) x^3 \cdot 2 - (4 - 6x^2) \cdot 6x^5}{x^9} e^{-1/x^2}$$

$$= \frac{x^3 (-12x) - (4 - 6x^2) \cdot 2 - (4 - 6x^2) \cdot 6x^2}{x^9} e^{-1/x^2}$$

$$E^{(n)}(x) = \frac{f_n(x)}{x^{3n}} e^{-1/x^2} \text{ when } x \neq 0, \text{ for some } f_n(x) \in \mathbb{R}[x].$$

We have seen this by computer for $n=0,1,2,3$.

How do we know the derivatives of E all have **this form**?

Proof For each $n \geq 0$ we have a statement about $E^{(n)}(x)$ having a given form.

Since $E(x) = e^{-1/x^2}$, the 0th derivative $E^{(0)}(x) = E(x)$ has the required form with $f_0(x) = 1$.

Now assuming $E^{(n)}(x) = \frac{f_n(x)}{x^{3n}} e^{-1/x^2} = \frac{f_n(x) e^{-1/x^2}}{x^{3n}}$, then

$$E^{(n+1)}(x) = \frac{x^{3n} \cdot (f_n'(x) e^{-1/x^2} + f_n(x) \cdot 2e^{-1/x^2} / x^3) - f_n(x) e^{-1/x^2} \cdot 3n x^{3n-1}}{x^{6n}}$$

$$= \frac{x^3 f_n'(x) + (2 - 3n x^2) f_n(x)}{x^{3(n+1)}} e^{-1/x^2}, \text{ for } x \neq 0$$

Note: $f_{n+1}(x) = x^3 f_n'(x) + (2 - 3n x^2) f_n(x) \in \mathbb{R}[x]$.

Next find $E^{(n)}(0)$ for $n=0, 1, 2, 3, \dots$, also by induction.

$$E^{(0)}(0) = E(0) = 0 \text{ by definition.}$$

Now assuming $E^{(n)}(0) = 0$ for some $n \geq 0$, we must conclude also $E^{(n+1)}(0) = 0$.

$$\begin{aligned} E^{(n+1)}(0) &= \lim_{h \rightarrow 0} \frac{E^{(n)}(h) - E^{(n)}(0)}{h} = \lim_{h \rightarrow 0} \frac{E^{(n)}(h)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f_n(h)}{h^{3n+1}} e^{-1/h^2} = f_n(0) \lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h^{3n+1}} \end{aligned}$$

Substituting $u = \frac{1}{h^2} \rightarrow \infty$ as $h \rightarrow 0$, $u = h^{-2}$, $h^2 = u^{-1}$, $h^{3n+1} = u^{-\frac{3n+1}{2}} = u^{-k}$

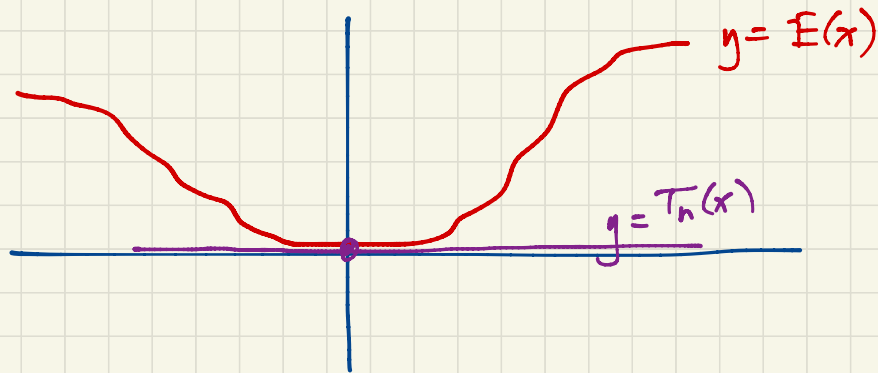
$$E^{(n+1)}(0) = f_n(0) \lim_{u \rightarrow \infty} \frac{e^{-u}}{u^{-k}} = \lim_{u \rightarrow \infty} \frac{u^k}{e^u} \text{ where } k = \frac{3n+1}{2} > 0.$$

$$\lim_{u \rightarrow \infty} \frac{u^k}{e^u} = \lim_{u \rightarrow \infty} \frac{k u^{k-1}}{e^u} = \lim_{u \rightarrow \infty} \frac{k(k-1) u^{k-2}}{e^u} = \dots = 0 \Rightarrow E^{(n+1)}(0) = f_n(0) \cdot 0 = 0.$$

$$E^{(n)}(x) = \begin{cases} \frac{f_n(x)}{x^{3n}} e^{-1/x^2}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

So the Taylor polynomial of $E(x)$ of order n centered at 0 is

$$T_n(x) = E(0) + E'(0)x + \frac{E''(0)}{2}x^2 + \frac{E'''(0)}{6}x^3 + \dots + \frac{E^{(n)}(0)}{n!}x^n = 0.$$



$\lim_{n \rightarrow \infty} T_n(x) \neq E(x)$
except at $x=0$.

$$T(x) = \sum_{n=0}^{\infty} \frac{E^{(n)}(0)}{n!} x^n$$

does not converge to $E(x)$.

$E(x)$ is not analytic at $x=0$.
(It is analytic everywhere else.)

Review: **Decimals** $\pi = 3.1415926\dots$

$$= 3 \times 10^0 + 1 \times 10^{-1} + 4 \times 10^{-2} + 1 \times 10^{-3} + 5 \times 10^{-4} + 9 \times 10^{-5} + \dots$$

$$100a = 1723.232323\dots$$

$$a = 17.232323\dots$$

$$99a = 1706$$

$$a = \frac{1706}{99} \in \mathbb{Q} \text{ and this fraction is reduced.}$$

Note: $a = 1 \times 10^1 + 7 \times 10^0 + 2 \times 10^{-1} + 3 \times 10^{-2} + 2 \times 10^{-3} + 3 \times 10^{-4} + \dots = \frac{1706}{99}$

Base 5 $a = 13.201201201\dots = 1 \times 5^1 + 3 \times 5^0 + 2 \times 5^{-1} + 0 \times 5^{-2} + 1 \times 5^{-3} + \dots$

$(125a) = 5^3 a = 13201.201201201\dots$
decimal $a = 13.201201201\dots$

$$a = \frac{1043}{124}$$

In decimal,

$$625 + 375 + 25 + 15 + 3 = 1043$$

Base 5 $13 \overset{1}{\underset{13}{\times}} \overset{5}{\underset{6}{0}} \overset{5}{\underset{13}{1}} \overset{5}{\underset{13}{2}} \overset{5}{\underset{13}{0}} \overset{5}{\underset{13}{1}} \dots$

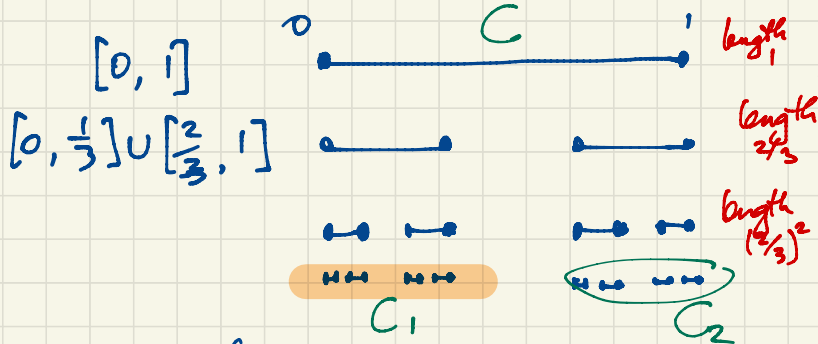
$$13133 = 1 \times 5^4 + 3 \times 5^3 + 1 \times 5^2 + 3 \times 5^1 + 3 \times 5^0$$

Cantor Set

Denote by C the Cantor set; and C_1 is the left half of the Cantor set.

$$C = C_1 \cup C_2$$

Scaling C_1 by a factor 3 gives $C = C_1 \cup C_2$ (two copies of C_1).

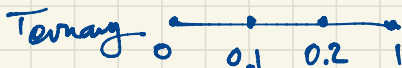
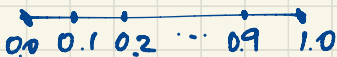


After infinitely many such steps, what's left is the Cantor set. $C = C_1 \cup C_2$. This set is totally disconnected (we'll define this later).

The Cantor set is actually all the points of $[0, 1]$ having a ternary expansion with only 0, 2 as its ternary digits.

Any number can be expressed in ternary using only 0, 1, 2 as digits.

Decimals



$$2 = 3^d$$

$$\Rightarrow \ln 2 = d \ln 3$$

$$\Rightarrow d = \frac{\ln 2}{\ln 3}$$

$$\approx 0.631$$

Length of Cantor set C is

$$\lim_{n \rightarrow \infty} \left(\frac{2}{3}\right)^n = 0$$

yet $|C| = |\mathbb{R}|$ (uncountable)

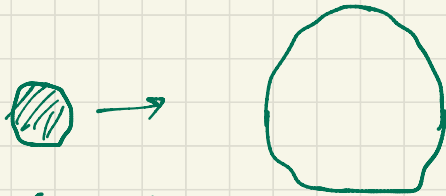
Dimension



has dimension 2.

Scaling our rectangle by a factor k in every direction results in k^2 copies of the original rectangle.

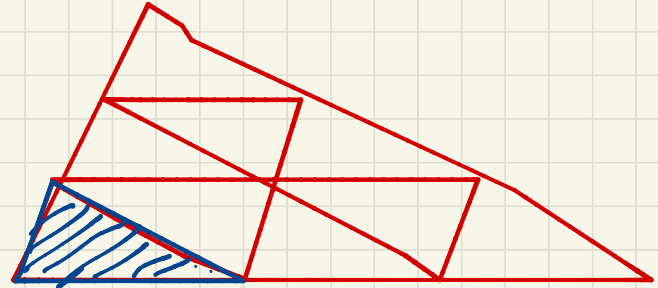
Same for disks



except a disk is not self-similar so the definition of dimension must be adapted for this situation.

Scaling by factor 3 in every direction results in 9 copies of the original region.

Same for a triangular region.



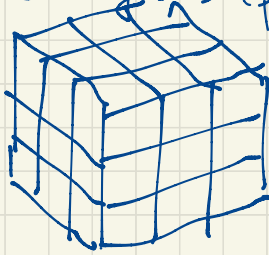
For self-similar objects (scaling by a factor of k gives k^d copies of the original) we call d the fractal dimension of the object.

For the rectangular and triangular regions (previous slide), $d=2$.



1x1x1 cube

scale by factor $k=3$



$3^3 = 27$ copies of the original cube.

so the cube is 3-dimensional. } $3 \times 3 \times 3$ cube

Intervals $[a, b] \subset \mathbb{R}$ are one-dimensional.



scale by factor $k=3$ in every direction



$3^1 = k = 3$ copies of the original

Similar to HW1 #3:

$$f(x) = \begin{cases} \frac{e^x - 1}{x}, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$$

Clearly f is continuous.

$$\text{For } x \neq 0, \quad f'(x) = \frac{x e^x - (e^x - 1) \cdot 1}{x^2} = \frac{(x-1)e^x + 1}{x^2}$$

$$f''(x) = \frac{x^2 [(x-1)e^x + e^x] - [(x-1)e^x + 1] 2x}{x^4}$$

$$= \frac{x [x e^x] - 2 [(x-1)e^x + 1]}{x^3}$$

$$= \frac{(x^2 - 2x + 2)e^x - 2}{x^3}$$

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{e^h - 1}{h} - 1}{h} \cdot \frac{h}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^h - 1 - h}{h^2} \\ &= \lim_{h \rightarrow 0} \frac{e^h - 1}{2h} \\ &= \lim_{h \rightarrow 0} \frac{e^h}{2} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0} f'(x) &= \lim_{x \rightarrow 0} \frac{(x-1)e^x + 1}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{e^x + (x-1)e^x}{2x} \\ &= \lim_{x \rightarrow 0} \frac{x e^x}{2x} = \lim_{x \rightarrow 0} \frac{e^x}{2} = \frac{1}{2} \end{aligned}$$

Note: f' is continuous i.e. $f \in C^1(\mathbb{R})$

$$\begin{aligned}
 f''(0) &= \lim_{h \rightarrow 0} \frac{f'(h) - f'(0)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{(h-1)e^h + 1}{h^2} - \frac{1}{2}}{h} \cdot \frac{h^2}{h^2} \\
 &= \lim_{h \rightarrow 0} \frac{(h-1)e^h + 1 - \frac{1}{2}h^2}{h^3} \\
 &= \lim_{h \rightarrow 0} \frac{e^h + (h-1)e^h - h}{3h^2} \\
 &= \lim_{h \rightarrow 0} \frac{he^h - h}{3h^2} \\
 &= \lim_{h \rightarrow 0} \frac{e^h - 1}{3h} \\
 &= \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \lim_{x \rightarrow 0} f''(x) &= \lim_{x \rightarrow 0} \frac{(x^2 - 2x + 2)e^x - 2}{x^3} \\
 &= \lim_{x \rightarrow 0} \frac{(x^2 - 2x + 2)e^x + (2x - 2)e^x}{3x^2} \\
 &= \lim_{x \rightarrow 0} \frac{x^2 e^x}{3x^2} \\
 &= \lim_{x \rightarrow 0} \frac{e^x}{3} = \frac{1}{3} = f''(0)
 \end{aligned}$$

So f'' is continuous. i.e. $f \in C^2(\mathbb{R})$.

Actually, we remark that $f \in C^\infty(\mathbb{R})$.

Its Taylor polynomial of degree 2 is

$$\begin{aligned}
 T_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \\
 &= 1 + \frac{1}{2}x + \frac{1}{6}x^2
 \end{aligned}$$

Check: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots$

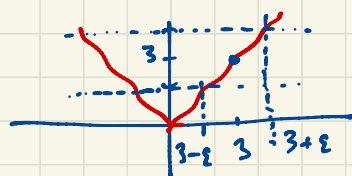
$$\frac{e^x - 1}{x} = \frac{x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots}{x} = 1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \dots$$

Similar to HW1 #2.

Prove that $\lim_{x \rightarrow 3} |x| = 3$.

Proof: Let $\varepsilon > 0$. Whenever $|x-3| < \varepsilon$, we have $||x|-3| \leq |x-3| < \varepsilon$.

Thus $\lim_{x \rightarrow 3} |x| = 3$. \square



It looks like $\delta = \varepsilon$ should work.

We want to show $|f(x)-3| < \varepsilon$ whenever $0 < |x-3| < \varepsilon$.

Recall: the triangle inequality

$$|x| - |y| \leq |x+y| \leq |x| + |y|$$

$$1-g(x) = \begin{cases} 1, & \text{if } x \notin \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{Q}. \end{cases}$$

$$\int_0^1 (1-g(x)) dx = \int_0^1 1 dx - \int_0^1 g(x) dx$$

p. 112 of Abbott: Dirichlet's function $g: \mathbb{R} \rightarrow \mathbb{R}$

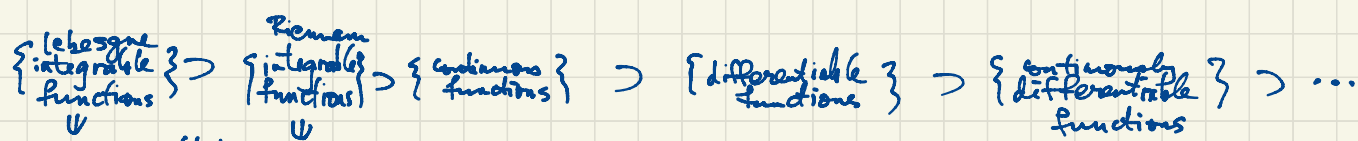
$$g(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

g is not continuous anywhere.

$\int_0^1 g(x) dx$ is undefined according to the Riemann definition of integral.

↑ This is the mean value of g on $[0, 1]$.

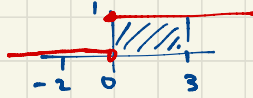
There are only countably many rationals in $[0, 1]$ but uncountably many irrationals in $[0, 1]$. So $\int_0^1 g(x) dx$ "should be" 0.



\Downarrow
 Dirichlet function

\Downarrow
 Heaviside function

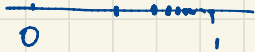
$$H(x) = \begin{cases} 1 & \text{if } x \geq 0; \\ 0 & \text{if } x < 0 \end{cases}$$



$$\int_{-2}^3 H(x) dx = 3$$

$S = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots\}$ is bounded: 0 is a lower bound, 1 is an upper bound.

$\frac{1}{2}$ is the greatest lower bound.



Every $m \leq \frac{1}{2}$ is a lower bound for S , meaning $s \geq \frac{1}{2}$ for all $s \in S$.

(so $\frac{1}{2}$ is a lower bound) and it is the greatest lower bound.

1 is the least upper bound of S .