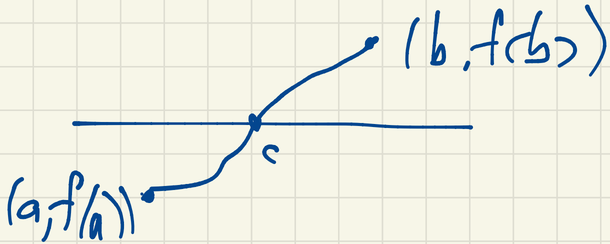
Analysis I (Math 3205)
Fall 2020

Book I

Intermediate Value Theorem

If $f : [a,b] \to \mathbb{R}$ is continuous with $f(a) < 0 < f(b)$, then there exists $c \in (a,b)$ such that $f(c) = 0$.

How does anyone prove this?
What is your experience with reading/writing proofs?



The theorem does not hold over $\mathbb{Q}$.

eg. $f(x) = x^2 - 2$, $f : \{\text{rationals between } 0 \text{ and } 2\} \to \mathbb{Q}$ is continuous, $f(0) < 0 < f(2)$ but there is no solution of $f(c) = 0$ in $\mathbb{Q}$.

$\mathbb{Q}$ is not complete; $\mathbb{R}$ is complete.

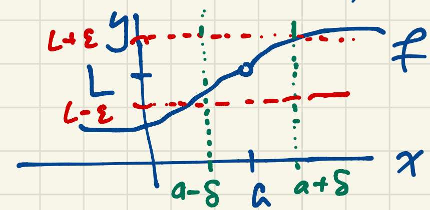
The complete statement of the Intermediate Value Theorem: For all $f : [a,b] \to \mathbb{R}$, if f is continuous and $f(a) < 0 < f(b)$, then there exists $c \in (a,b)$ such that $f(c) = 0$.

"for all", "for every", "for each" : Universal quantifiers

"there is", "there exists" : existential quantifiers.

For all x there exists y such that $x < y$. (True in $\mathbb{R}$)

There exists y such that for all x, $x < y$. (False in $\mathbb{R}$)



Definition of a Limit

We say $\lim_{x \to a} f(x) = L$ if the following condition holds:

For all $\varepsilon > 0$, there exists $\delta > 0$ such that

$\underbrace{|f(x) - L| < \varepsilon}_{f(x) \text{ is within } \varepsilon \text{ of } L}$ whenever $\underbrace{0 < |x-a| < \delta}_{x \text{ is within } \delta \text{ of } a}$.

Note: It doesn't matter here what $f(a)$ is or even whether or not it's defined

i.e. for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all x, if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

Let's prove that $\lim_{x\to 2}(5x+1) = 11$.

Rough version: $f(x) = 5x+1$. If we need $f(x)$ to be within ε of 11, how close does x have to be to 2?

$$|f(x) - 11| < \varepsilon \iff 11-\varepsilon < f(x) < 11+\varepsilon$$
$$\iff 11-\varepsilon < 5x+1 < 11+\varepsilon$$
$$\iff 10-\varepsilon < 5x < 10+\varepsilon$$
$$\iff 2-\frac{\varepsilon}{5} < x < 2+\frac{\varepsilon}{5}$$
$$\iff |x-2| < \frac{\varepsilon}{5}$$

Proof (Actually): Let $\varepsilon > 0$. Then whenever $0 < |x-2| < \frac{\varepsilon}{5}$ we have $2-\frac{\varepsilon}{5} < x < 2+\frac{\varepsilon}{5}$ so

$11-\varepsilon < 5x+1 < 11+\varepsilon$ i.e. $|f(x) - 11| < \varepsilon$.

Another proof: Suppose $\lim_{x\to 7} f(x) = 4$ and $\lim_{x\to 7} g(x) = 5$. Prove that $\lim_{x\to 7}(f(x)+g(x)) = 9$.

Rough version: Given $\varepsilon > 0$ we must find $\delta > 0$ such that $|f(x)+g(x) - 9| < \varepsilon$ whenever $0 < |x-7| < \delta$. Since $\lim_{x\to 7} f(x) = 4$, we can find $\delta > 0$ such that $|f(x) - 4| < \varepsilon$ whenever $0 < |x-7| < \delta$. Also since $\lim_{x\to 7} g(x) = 5$, we can find δ' such that

$|g(x) - 7| < \varepsilon$ whenever $0 < |x-7| < \delta'$.

$4-\varepsilon < f(x) < 4+\varepsilon$ whenever $7-\delta < x < 7+\delta$

$4-\varepsilon < f(x) < 4+\varepsilon$ whenever $7-\delta < x < 7+\delta$ i.e. $|x-7| < \delta$

$5-\varepsilon < g(x) < 5+\varepsilon$ whenever $7-\delta' < x < 7+\delta'$ i.e. $|x-7| < \delta'$

$$\underbrace{9-2\varepsilon < f(x)+g(x) < 9+2\varepsilon} \text{ whenever } 0<|x-7| < \min\{\delta, \delta'\}$$

$$|f(x)+g(x)-9| < 2\varepsilon \quad \text{whenever} \quad 0<|x-7| < \min\{\delta, \delta'\}.$$

Actual (final) proof:

Let $\varepsilon > 0$. There exists δ such that $|f(x)-4| < \frac{\varepsilon}{2}$ whenever $0<|x-7|<\delta$. Also there exists $\delta' > 0$ such that $|g(x)-5| < \frac{\varepsilon}{2}$ whenever $0<|x-7|<\delta'$. Then

$$|f(x)+g(x)-9| \le |f(x)-4| + |g(x)-5| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $0 < |x-7| < \min\{\delta, \delta'\}$. $\square$

Note: The triangle inequality says $|a+b| \le |a| + |b|$ for all a, b.

$|f(x)+g(x)-9| = |f(x)-4+g(x)-5|$

$|f(x)-4+g(x)-5| \le |f(x)-4| + |g(x)-5|$

$|f(x)+g(x)-9| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$|f(x)+g(x)-9| < \varepsilon$

$7-2=5 \iff 7-2=\boxed{5}$

$\iff 7=\boxed{5}+2$

$$\left.\begin{array}{l}\infty+1=\infty \iff \infty-\infty=1 \\ \infty+2=\infty \iff \infty-\infty=2\end{array}\right\} \Rightarrow 1=2.$$

$\lim_{x\to\infty} f(x)=\infty$ <u>means</u> for all M there exist N such that $f(x)>M$ whenever $x>N$.

$[N,\infty)=\{a\in\mathbb{R} : a\geq N\}.$

$(N,\infty)=\{a\in\mathbb{R} : a>N\}.$

$0.9999999\ldots = 1$

$\frac{1}{3}=0.333333\ldots$

Eg. $g(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$

For $x \neq 0$, $\quad -1 \leq \sin\frac{1}{x} \leq 1$

so $\quad -x^2 \leq \underbrace{x^2 \sin\frac{1}{x}}_{g(x)} \leq x^2$

$$-x^2 \to 0, \quad g(x) \to 0, \quad x^2 \to 0$$

Why is g continuous at 0?

$\lim_{x \to 0} g(x) = 0$ because $-x^2 \leq g(x) \leq x^2$

where $\lim_{x \to 0} (-x^2) = 0 = \lim_{x \to 0} x^2$ so we can use the Squeeze Theorem.

Since $\lim_{x \to 0} g(x) = 0 = g(0)$, g is continuous at 0.

Is g differentiable at 0?

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} h \sin\frac{1}{h} = 0$$

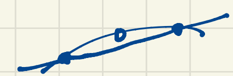
by the Squeeze Theorem since $-|h| \leq h \sin\frac{1}{h} \leq |h|$ whenever $h \neq 0$

where $\lim_{h \to 0} |h| = 0 = \lim_{h \to 0} (-|h|)$.

$$-1 \leq \sin\frac{1}{h} \leq 1$$

$$-|h| \leq h \sin\frac{1}{h} \leq |h|$$

$\lim_{h \to 0} \frac{g(a+h) - g(a-h)}{2h}$? slope of secant line between $(a-h, g(a-h))$ and $(a+h, g(a+h))$



$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

$$g(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

$$g'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Uses the chain rule and the product rule

If $x \neq 0$, $g'(x) = x^2 \left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + 2x \sin\frac{1}{x} = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$

Note: g is differentiable (i.e. everywhere on its domain which is $\mathbb{R}$).

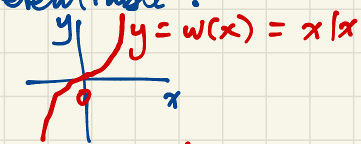
It is not possible to evaluate $g'(0)$ using the chain rule, product rule, rules for derivatives of power functions and trig functions, etc.

$\lim\limits_{x\to 0} g'(x)$ does not exist. So g' is not continuous at 0.

g is differentiable but not continuously differentiable.

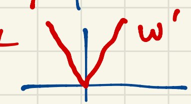
Example of a function which is differentiable but not twice differentiable:

$g(x)$ as above; also $w(x) = x|x| = \begin{cases} x^2, & \text{if } x \geq 0; \\ -x^2, & \text{if } x < 0. \end{cases}$



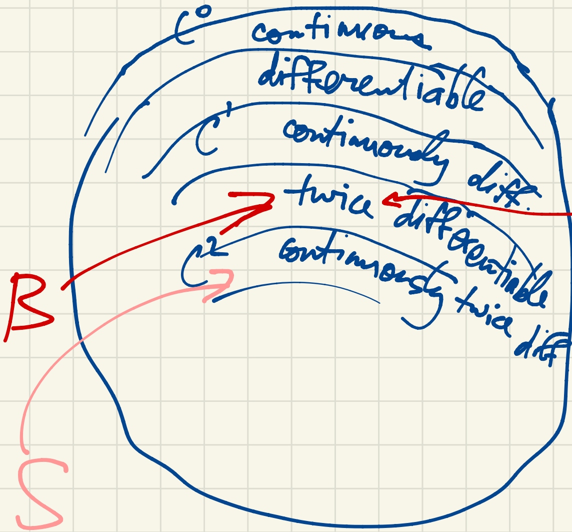
Check: $w'(x) = 2|x|$

$w''(0)$ does not exist. w is continuously differentiable but not twice differentiable.



Hierarchy of Smoothness:

$C^n = C^n(\mathbb{R})$ is the set of functions $\mathbb{R} \to \mathbb{R}$ which are continuously n times differentiable.



What is an example of a function that is twice differentiable but not continuously twice differentiable? Find $f: \mathbb{R} \to \mathbb{R}$ such that f'' is defined everywhere but f'' is not continuous.

$$f(x) = \begin{cases} x(\ln|x| - 1), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$\lim_{x\to 0} f(x) = \lim_{x \to 0} x(\ln|x| - 1) = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x(\ln x - 1) = \lim_{x\to 0^+} \frac{\ln x - 1}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x\to 0^+} (-x) = 0$$

f is now continuous; is it differentiable?

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \frac{h(\ln|h| - 1) - 0}{h} = \lim_{h\to 0} (\ln|h| - 1)$$
$$= -\infty$$

so this function is not differentiable.

$V(x) = \sin \frac{1}{x^2+1}$ is C^∞ i.e. $V \in C^\infty(\mathbb{R})$ (this function is infinitely differentiable on all of $\mathbb{R}$)

$$V'(x) = \cos \frac{1}{x^2+1} \cdot \left(\frac{(x^2+1)\cdot 0 - 1 \cdot 2x}{(x^2+1)^2} \right) = \frac{-2x}{(1+x^2)^2} \cos \frac{1}{1+x^2}$$

If we continue taking higher and higher order derivatives, every $V^{(n)}(x)$ has terms of the form $\frac{\text{polynomial in } x}{(x^2+1)^k} \cdot \sin\left(\frac{1}{x^2+1}\right)$

or cos

$$R(x) = \begin{cases} x^3 \sin \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

If $x \neq 0$, $R'(x) = 3x^2 \sin\frac{1}{x} + x^3 \cdot \left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}$.

$$R'(0) = \lim_{h\to 0} \frac{R(h) - R(0)}{h} = \lim_{h\to 0} \frac{h^3 \sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} h^2 \sin\frac{1}{h} = 0$$

(see Tuesday's class)

$$R'(x) = \begin{cases} 3x^2 \sin\frac{1}{x} - x\cos\frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

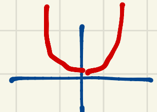
$$R''(x) = 6x\sin\frac{1}{x} + 3x^2 \cdot \cos\frac{1}{x} \cdot \left(\frac{-1}{x^2}\right) - \cos\frac{1}{x} + x\sin\frac{1}{x}\left(\frac{-1}{x^2}\right)$$

$$= 6x\sin\frac{1}{x} - 4\cos\frac{1}{x} - \frac{1}{x}\sin\frac{1}{x}, \quad \text{if } x \neq 0.$$

$$R''(0) = \lim_{h\to 0} \frac{R'(h) - R'(0)}{h} = \lim_{h\to 0} \frac{3h^2\sin\frac{1}{h} - h\cos\frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0} \left(3h\sin\frac{1}{h} - \cos\frac{1}{h}\right) \text{ does not exist.}$$

So R is not twice differentiable.

Try $S(x) = x^2|x|$ 

$$S'(x) = \lim_{h\to 0} \frac{S(x+h) - S(x)}{h}$$

If $x>0$, $S(x) = x^3$, $S'(x) = 3x^2$, $S''(x) = 6x$.

If $x<0$, $S(x) = -x^3$, $S'(x) = -3x^2$, $S''(x) = -6x$.

$$S'(0) = \lim_{h\to 0} \frac{h^2|h| - 0}{h} = \lim_{h\to 0} h|h| = 0.$$

Note that $S'(x) = 3x|x|$ for all x. So S' is continuous i.e. S is continuously differentiable $(S \in C^1(\mathbb{R}))$.

$$S''(0) = \lim_{h\to 0} \frac{S'(h) - S'(0)}{h} = \lim_{h\to 0} \frac{3h|h| - 0}{h} = \lim_{h\to 0} 3|h| = 0$$

Note: $S''(x) = 6|x|$ for all x

Try $B(x) = \begin{cases} x^4 \sin\frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0 \end{cases}$

$B''(x)$ is defined for all x but not continuous at 0. since $\lim_{x\to 0} B''(x)$ does not exist.

For $x \neq 0$, $B'(x) = 4x^3 \sin\frac{1}{x} + x^4 \cdot \left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = 4x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x}$

$$B''(x) = 12x^2 \sin\frac{1}{x} + 4x^3 \cos\frac{1}{x}\left(-\frac{1}{x^2}\right) - 2x\cos\frac{1}{x} - x^2\left(-\sin\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$$

$$= (12x^2 - 1)\sin\frac{1}{x} - 6x\cos\frac{1}{x}$$

$$B'(0) = \lim_{h\to 0} \frac{B(h) - B(0)}{h} = \lim_{h\to 0} \frac{h^4 \sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} h^3 \sin\frac{1}{h} = 0$$

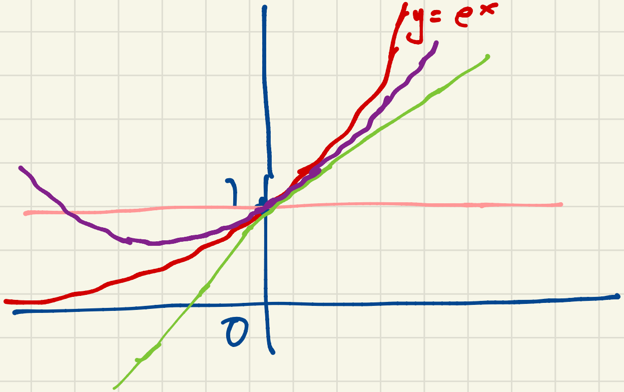
Note: $\lim_{x\to 0} B'(x) = \lim_{x\to 0} \left(4x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x}\right) = 0 = B'(0)$ so B' is continuous

i.e. $B \in C^1(\mathbb{R})$ i.e. B is continuously differentiable

$$B''(0) = \lim_{h\to 0} \frac{B'(h) - B'(0)}{h}$$

$$= \lim_{h\to 0} \frac{4h^3 \sin\frac{1}{h} - h^2\cos\frac{1}{h} - 0}{h} = \lim_{h\to 0}\left(4h^2 \sin\frac{1}{h} - h\cos\frac{1}{h}\right) = 0$$

$\lim_{x\to 0} B''(x) = \lim_{x\to 0}\left((\cancel{12x^2}-1)\sin\frac{1}{x} - 6x\cos\frac{1}{x}\right)$ does not exist!



Crude (0^{th} order approximation):

$e^x \simeq 1$ for $x \approx 0$.

Tangent line approximation: (1^{st} order approximation)

$e^x \approx 1+x$ for $x \approx 0$.

Quadratic approximation (2^{nd} order approximation):

$e^x \approx 1+x+\frac{x^2}{2}$ for $x \approx 0$

Taylor polynomial $T_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$

is the "best" polynomial approximation of degree n to the function e^x

For a general function $f \in C^n$, the Taylor polynomial of degree n is

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$n! = 1 \times 2 \times 3 \times \cdots \times n$.

Eg. $f(x) = \dfrac{x}{1+2x}$

$f(0) = 0$

$f'(0) = 1$

$f''(0) = -4$

$f'''(0) = 24$

etc.

$$f'(x) = \frac{(1+2x)\cdot 1 - x\cdot 2}{(1+2x)^2} = \frac{1}{(1+2x)^2}$$

$$f''(x) = \frac{\cancel{(1+2x)^2\cdot 0} - 1\cdot 2(1+2x)\cdot 2}{(1+2x)^4} = \frac{-4}{(1+2x)^3}$$

$$f'''(x) = \frac{\cancel{(1+2x)^3\cdot 0} - (-4)\cdot 3(1+2x)^2\cdot 2}{(1+2x)^6} = \frac{24}{(1+2x)^4}$$

The Taylor series for f centred at 0 is

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \cdots$$

$$= 0 + x - \frac{4}{2}x^2 + \frac{24}{6}x^3 + \cdots$$

$$= x - 2x^2 + 4x^3 - 8x^4 + 16x^5 - 32x^6 + \cdots$$

Note: $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$ (geometric series)

$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\cdots$ (geometric series) This converges for $|x|<1$.

$\frac{1}{1+2x} = 1-2x+4x^2-8x^3+16x^4-32x^5+\cdots$ (converges for $|-2x|<1$ i.e. $|x|<\frac{1}{2}$)

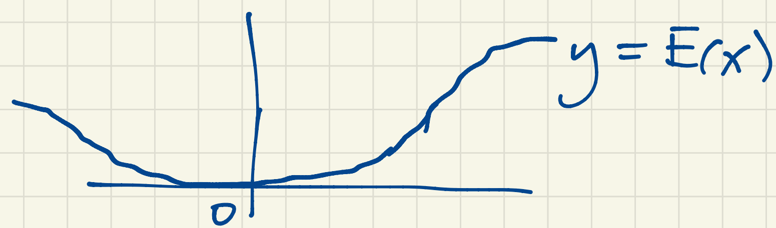
$\frac{x}{1+2x} = x-2x^2+4x^3-8x^4+16x^5-32x^6+\cdots$ (converges for $|x|<\frac{1}{2}$)

By the way, where does this series converge? (Review from Calc II)
Ratio Test.

Another example: $x^2\sin x = x^2\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\cdots\right)$

$= x^3-\frac{x^5}{6}+\frac{x^7}{120}-\frac{x^9}{5040}+\cdots$ for all x.

Eg. $E(x) = \begin{cases} e^{-1/x^2}, & \text{if } x\neq 0; \\ 0, & \text{if } x=0. \end{cases}$



$$E(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

$$\frac{d}{dx}\left(-\frac{1}{x^2}\right) = \frac{d}{dx}\left(-x^{-2}\right) = 2x^{-3}$$

If $x \neq 0$, $E'(x) = e^{-1/x^2}\left(2x^{-3}\right) = 2x^{-3} e^{-1/x^2} = 2\,\frac{e^{-1/x^2}}{x^3}$

$$E''(x) = 2\left(\frac{x^3 \cdot 2e^{-1/x^2}/x^3 - e^{-1/x^2} \cdot 3x^2}{x^6}\right)$$

$$= \frac{(4-6x^2)\,e^{-1/x^2}}{x^6}$$

$$E'''(x) = \frac{x^6 \cdot \left[(-12x)e^{-1/x^2} + (4-6x^2)\left(2e^{-1/x^2}/x^3\right)\right] - (4-6x^2)e^{-1/x^2} \cdot 6x^5}{x^{12}}$$

$$= \frac{x^6(-12x) + (4-6x^2)\,x^3 \cdot 2 - (4-6x^2) \cdot 6x^5}{x^{12}}\, e^{-1/x^2}$$

$$= \frac{x^3(-12x) - (4-6x^2) \cdot 2 - (4-6x^2) \cdot 6x^2}{x^9}\, e^{-1/x^2}$$

$$E^{(n)}(x) = \frac{f_n(x)}{x^{3n}} e^{-1/x^2} \text{ when } x \neq 0, \text{ for some } f_n(x) \in \mathbb{R}[x].$$

We have seen this by computer for $n = 0, 1, 2, 3$.

How do we know the derivatives of E all have this form?

<u>Proof</u> For each $n \geq 0$ we have a statement about $E^{(n)}(x)$ having a given form.

Since $E(x) = e^{-1/x^2}$, the 0^{th} derivative $E^{(0)}(x) = E(x)$ has the required form with $f_0(x) = 1$.

Now assuming $E^{(n)}(x) = \frac{f_n(x)}{x^{3n}} e^{-1/x^2} = \frac{f_n(x) e^{-1/x^2}}{x^{3n}}$, then

$$E^{(n+1)}(x) = \frac{x^{3n} \cdot \left(f_n'(x) e^{-1/x^2} + f_n(x) \cdot 2e^{-1/x^2}/x^3 \right) - f_n(x) e^{-1/x^2} \cdot 3n x^{3n-1}}{x^{6n}}$$

$$= \frac{x^3 f_n'(x) + (2 - 3n x^2) f_n(x)}{x^{3(n+1)}} e^{-1/x^2}, \text{ for } x \neq 0$$

Note: $f_{n+1}(x) = x^3 f_n'(x) + (2 - 3n x^2) f_n(x) \in \mathbb{R}[x]$.

Next find $E^{(n)}(0)$ for $n = 0, 1, 2, 3, \dots$, also by induction.

$E^{(0)}(0) = E(0) = 0$ by definition.

Now assuming $E^{(n)}(0) = 0$ for some $n \geq 0$, we must conclude also $E^{(n+1)}(0) = 0$.

$$E^{(n+1)}(0) = \lim_{h \to 0} \frac{E^{(n)}(h) - E^{(n)}(0)}{h} = \lim_{h \to 0} \frac{E^{(n)}(h)}{h}$$

$$= \lim_{h \to 0} \frac{f_n(h)}{h^{3n+1}} e^{-1/h^2} = f_n(0) \lim_{h \to 0} \frac{e^{-1/h^2}}{h^{3n+1}}$$

Substituting $u = \frac{1}{h^2} \to \infty$ as $h \to 0$, $u = h^{-2}$, $h^2 = u^{-1}$, $h^{3n+1} = u^{-\frac{3n+1}{2}}$
$= u^{-k}$

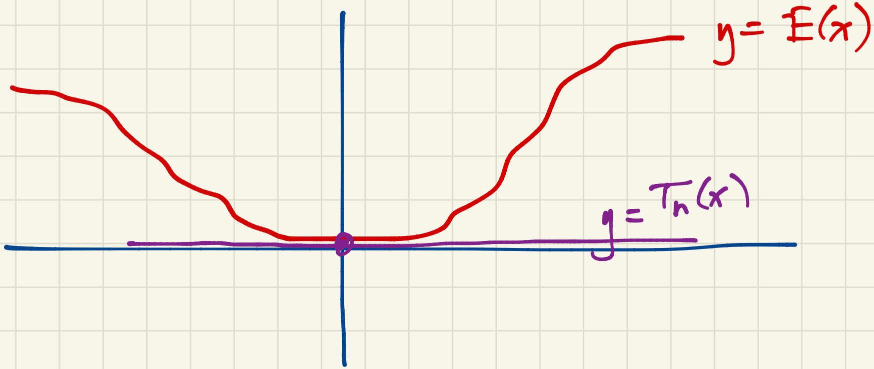
$$E^{(n+1)}(0) = f_n(0) \lim_{u \to \infty} \frac{e^{-u}}{u^{-k}} = f_n(0) \lim_{u \to \infty} \frac{u^k}{e^u} \quad \text{where } k = \frac{3n+1}{2} > 0.$$

$$\lim_{u \to \infty} \frac{u^k}{e^u} = \lim_{u \to \infty} \frac{k u^{k-1}}{e^u} = \lim_{u \to \infty} \frac{k(k-1) u^{k-2}}{e^u} = \dots = 0 \implies E^{(n+1)}(0) = f_n(0) \cdot 0 = 0.$$

$$E^{(n)}(x) = \begin{cases} \dfrac{f_n(x)}{x^{3n}} e^{-1/x^2}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

So the Taylor polynomial of $E(x)$ of order n centered at 0 is

$$T_n(x) = E(0) + E'(0)x + \frac{E''(0)}{2}x^2 + \frac{E'''(0)}{6}x^2 + \dots \frac{E^{(n)}(0)}{n!}x^n$$
$$= 0.$$



$\lim\limits_{n\to\infty} T_n(x) \neq E(x)$ except at $x=0$.

$T(x) = \sum\limits_{n=0}^{\infty} \dfrac{E^{(n)}(0)}{n!}x^n$

does not converge to $E(x)$.

$E(x)$ is not analytic at $x=0$.

(It is analytic everywhere else.)

Review: **Decimals** $\pi = 3.1415926\ldots$

$$= 3\times10^0 + 1\times10^{-1} + 4\times10^{-2} + 1\times10^{-3} + 5\times10^{-4} + 9\times10^{-5} + \cdots$$

$$100a = 1723.232323\ldots$$
$$a = 17.232323\ldots$$

$$99a = 1706$$

$a = \frac{1706}{99} \in \mathbb{Q}$ and this fraction is reduced.

Note: $a = 1\times10^1 + 7\times10^0 + 2\times10^{-1} + 3\times10^{-2} + 2\times10^{-3} + 3\times10^{-4} + \cdots = \frac{1706}{99}$.

Base 5 $a = 13.201201201\ldots = 1\times5^1 + 3\times5^0 + 2\times5^{-1} + 0\times5^{-2} + 1\times5^{-3} + \cdots$

$$\underbrace{125a}_{\text{decimal}} = 5^3 a = 13201.201201201\ldots$$
$$a = 13.201201201\ldots$$

$$a = \frac{1043}{124}$$

Base 5

$$\begin{array}{r} 13201 \\ -\ 13 \\ \hline 13133 \end{array} = 1\times5^4 + 3\times5^3 + 1\times5^2 + 3\times5^1 + 3\times5^0$$

In decimal,

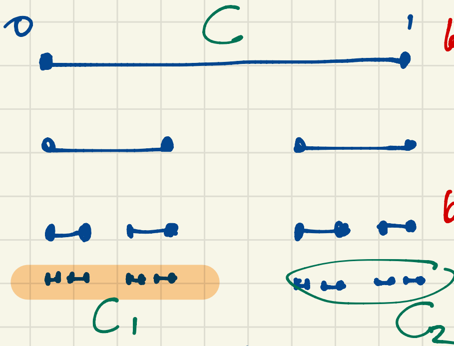
$$625 + 375 + 25 + 15 + 3 = 1043$$

Cantor Set

Denote by C the Cantor set; and C_1 is the left half of the Cantor set.

$[0,1]$

$[0,\frac{1}{3}] \cup [\frac{2}{3}, 1]$



length 1

length $2/3$

length $(2/3)^2$

$$C = C_1 \sqcup C_2$$

Scaling C_1 by a factor 3 gives $C = C_1 \sqcup C_2$ (two copies of C_1).

$[0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$

$[0,\frac{1}{27}] \cup [\frac{2}{27},\frac{1}{9}] \cup \dots \cup [\frac{26}{27},1]$

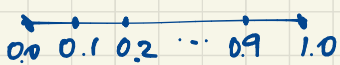
After infinitely many such steps, what's left is the Cantor set. $C = C_1 \sqcup C_2$

This set is totally disconnected (we'll define this later).

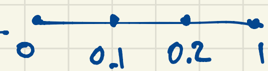
The Cantor set is actually all the points of $[0,1]$ having a ternary expansion with only 0, 2 as its ternary digits.

Any number can be expressed in ternary using only 0, 1, 2 as digits.

Decimals



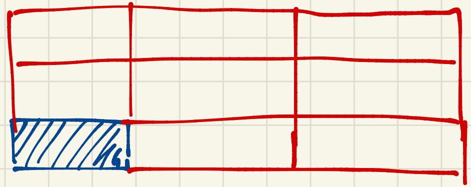
Ternary



$$2 = 3^d \implies \ln 2 = d \ln 3 \implies d = \frac{\ln 2}{\ln 3} \approx 0.631$$

Length of Cantor set C is $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$. Yet $|C| = |\mathbb{R}|$ (uncountable)

Dimension

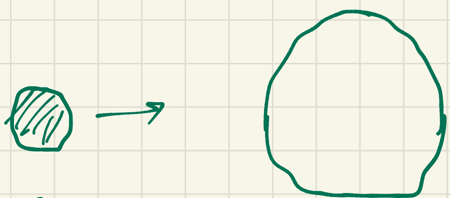


has dimension 2

Scaling by factor 3 in every direction results in 9 copies of the original region.

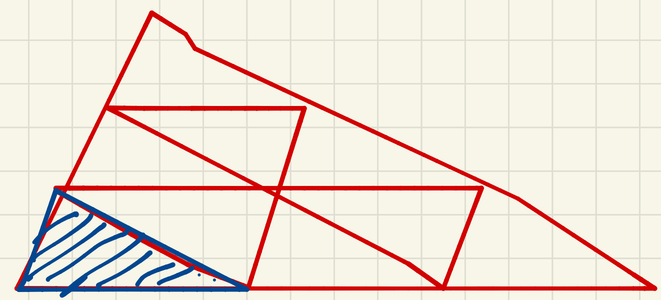
Scaling our rectangle by a factor k in every direction results in k^2 copies of the original rectangle.

Same for disks



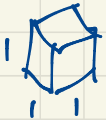
except a disk is not self-similar so the definition of dimension must be adapted for this situation.

Same for a triangular region.



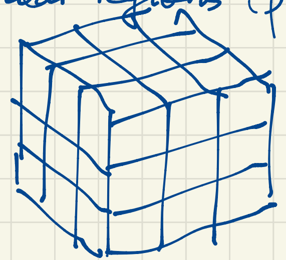
For self-similar objects (scaling by a factor of k gives k^d copies of the original) we call d the **fractal dimension** of the object.

For the rectangular and triangular regions (previous slide), $d=2$.



$1 \times 1 \times 1$ cube

$\xrightarrow{\text{scale by factor } k=3}$



$3 \times 3 \times 3$ cube

$3^3 = 27$ copies of the original cube.

so the cube is 3-dimensional.

Intervals $[a,b] \subset \mathbb{R}$ are one-dimensional.

$\xrightarrow{}$

scale by factor $k=3$ in every direction

$3^1 = k = 3$ copies of the original

Similar to HW1 #3:

$$f(x) = \begin{cases} \frac{e^x - 1}{x}, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$$

Clearly f is continuous.

For $x \neq 0$, $f'(x) = \frac{xe^x - (e^x - 1)\cdot 1}{x^2} = \frac{(x-1)e^x + 1}{x^2}$

$$f''(x) = \frac{x^2[(x-1)e^x + e^x] - [(x-1)e^x + 1]2x}{x^4}$$

$$= \frac{x[xe^x] - 2[(x-1)e^x + 1]}{x^3}$$

$$= \frac{(x^2 - 2x + 2)e^x - 2}{x^3}$$

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{e^h - 1}{h} - 1}{h} \cdot \frac{h}{h}$$

$$= \lim_{h \to 0} \frac{e^h - 1 - h}{h^2}$$

$$= \lim_{h \to 0} \frac{e^h - 1}{2h}$$

$$= \lim_{h \to 0} \frac{e^h}{2} = \frac{1}{2}.$$

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} \frac{(x-1)e^x + 1}{x^2}$$

$$= \lim_{x \to 0} \frac{e^x + (x-1)e^x}{2x}$$

$$= \lim_{x \to 0} \frac{xe^x}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}$$

Note: f' is continuous i.e. $f \in C^1(\mathbb{R})$

$$f''(0) = \lim_{h\to 0} \frac{f'(h) - f'(0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(h-1)e^h + 1}{h^2} - \frac{1}{2}}{h} \cdot \frac{h^2}{h^2}$$

$$= \lim_{h\to 0} \frac{(h-1)e^h + 1 - \frac{1}{2}h^2}{h^3}$$

$$= \lim_{h\to 0} \frac{e^h + (h-1)e^h - h}{3h^2}$$

$$= \lim_{h\to 0} \frac{he^h - h}{3h^2}$$

$$= \lim_{h\to 0} \frac{e^h - 1}{3h}$$

$$= \lim_{h\to 0} \frac{e^h}{3} = \frac{1}{3}.$$

$$\lim_{x\to 0} f''(x) = \lim_{x\to 0} \frac{(x^2 - 2x + 2)e^x - 2}{x^3}$$

$$= \lim_{x\to 0} \frac{(x^2-2x+2)e^x + (2x-2)e^x}{3x^2}$$

$$= \lim_{x\to 0} \frac{x^2 e^x}{3x^2}$$

$$= \lim_{x\to 0} \frac{e^x}{3} = \frac{1}{3} = f''(0)$$

So f'' is continuous. i.e. $f \in C^2(\mathbb{R})$.

Actually, we remark that $f \in C^\infty(\mathbb{R})$.

It's Taylor polynomial of degree 2 is

$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

$$= 1 + \frac{1}{2}x + \frac{1}{6}x^2.$$

Check: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$

$$\frac{e^x - 1}{x} = \frac{x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots}{x} = 1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \cdots$$