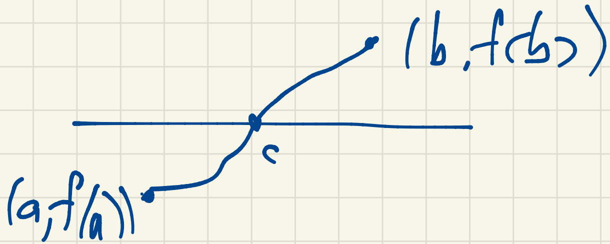
Analysis I (Math 3205)
Fall 2020

Book I

Intermediate Value Theorem

If $f: [a,b] \to \mathbb{R}$ is continuous with $f(a) < 0 < f(b)$, then there exists $c \in (a,b)$ such that $f(c) = 0$.

How does anyone prove this?
what is your experience with reading/writing proofs?



The theorem does not hold over $\mathbb{Q}$.

eg. $f(x) = x^2 - 2$, f: {rationals between 0 and 2} $\to \mathbb{Q}$ is continuous, $f(0) < 0 < f(2)$ but there is no solution of $f(c) = 0$ in $\mathbb{Q}$.

$\mathbb{Q}$ is not complete; $\mathbb{R}$ is complete.

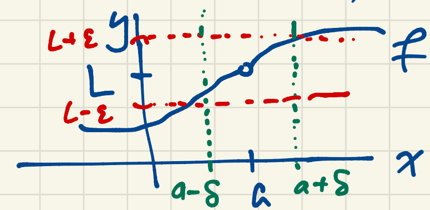
The complete statement of the Intermediate Value Theorem: For all $f: [a,b] \to \mathbb{R}$, if f is continuous and $f(a) < 0 < f(b)$, then there exists $c \in (a,b)$ such that $f(c) = 0$.

"for all", "for every", "for each" : Universal quantifiers

"there is", "there exists" : existential quantifiers.

For all x there exists y such that $x < y$. (True in $\mathbb{R}$)

There exists y such that for all x, $x < y$. (False in $\mathbb{R}$)



Definition of a Limit

We say $\lim_{x\to a} f(x) = L$ if the following condition holds:

For all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{|f(x) - L| < \varepsilon}_{f(x) \text{ is within } \varepsilon \text{ of } L} \quad \text{whenever} \quad \underbrace{0 < |x-a| < \delta}_{x \text{ is within } \delta \text{ of } a}.$$

Note: It doesn't matter here what $f(a)$ is or even whether or not it's defined

i.e. for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all x, if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

Let's prove that $\lim_{x\to 2}(5x+1) = 11$.

Rough version: $f(x) = 5x+1$. If we need $f(x)$ to be within ε of 11, how close does x have to be to 2?

$$|f(x) - 11| < \varepsilon \iff 11-\varepsilon < f(x) < 11+\varepsilon$$
$$\iff 11-\varepsilon < 5x+1 < 11+\varepsilon$$
$$\iff 10-\varepsilon < 5x < 10+\varepsilon$$
$$\iff 2-\frac{\varepsilon}{5} < x < 2+\frac{\varepsilon}{5}$$
$$\iff |x-2| < \frac{\varepsilon}{5}$$

Proof (Actually): Let $\varepsilon > 0$. Then whenever $0 < |x-2| < \frac{\varepsilon}{5}$ we have $2-\frac{\varepsilon}{5} < x < 2+\frac{\varepsilon}{5}$ so

$11-\varepsilon < 5x+1 < 11+\varepsilon$ i.e. $|f(x) - 11| < \varepsilon$.

Another proof: Suppose $\lim_{x\to 7} f(x) = 4$ and $\lim_{x\to 7} g(x) = 5$. Prove that $\lim_{x\to 7}(f(x)+g(x)) = 9$.

Rough version: Given $\varepsilon > 0$ we must find $\delta > 0$ such that $|f(x)+g(x)-9| < \varepsilon$ whenever $0 < |x-7| < \delta$. Since $\lim_{x\to 7} f(x) = 4$, we can find $\delta > 0$ such that $|f(x)-4| < \varepsilon$ whenever $0 < |x-7| < \delta$. Also since $\lim_{x\to 7} g(x) = 5$, we can find δ' such that

$|g(x) - 7| < \varepsilon$ whenever $0 < |x-7| < \delta'$.

$4-\varepsilon < f(x) < 4+\varepsilon$ whenever $7-\delta < x < 7+\delta$

$4-\varepsilon < f(x) < 4+\varepsilon$ whenever $7-\delta < x < 7+\delta$ i.e. $|x-7| < \delta$

$5-\varepsilon < g(x) < 5+\varepsilon$ whenever $7-\delta' < x < 7+\delta'$ i.e. $|x-7| < \delta'$

$$\underbrace{9-2\varepsilon < f(x)+g(x) < 9+2\varepsilon}\text{ whenever } 0<|x-7| < \min\{\delta, \delta'\}$$

$$|f(x)+g(x)-9| < 2\varepsilon \quad \text{whenever} \quad 0<|x-7| < \min\{\delta,\delta'\}.$$

Actual (final) proof:

Let $\varepsilon > 0$. There exists δ such that $|f(x)-4| < \frac{\varepsilon}{2}$ whenever $0<|x-7|<\delta$. Also there exists $\delta' > 0$ such that $|g(x)-5| < \frac{\varepsilon}{2}$ whenever $0<|x-7| < \delta'$. Then

$$|f(x)+g(x)-9| \leq |f(x)-4| + |g(x)-5| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $0 < |x-7| < \min\{\delta, \delta'\}$. $\square$

Note: The triangle inequality says $|a+b| \leq |a| + |b|$ for all a, b.

$$|f(x)+g(x)-9| = |f(x)-4+g(x)-5|$$

$$|f(x)-4+g(x)-5| \leq |f(x)-4| + |g(x)-5|$$

$$|f(x)+g(x)-9| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$|f(x)+g(x)-9| < \varepsilon$$

$7-2=5 \iff 7-2=\boxed{5}$

$\iff 7=\boxed{5}+2$

$$\left.\begin{array}{l} \infty+1=\infty \iff \infty-\infty=1 \\ \infty+2=\infty \iff \infty-\infty=2 \end{array}\right\} \Rightarrow 1=2.$$

$\lim\limits_{x\to\infty} f(x)=\infty$ <u>means</u> for all M there exist N such that $f(x)>M$ whenever $x>N$.

$[N,\infty)=\{a\in\mathbb{R} : a\geq N\}$.

$(N,\infty)=\{a\in\mathbb{R} : a>N\}$.

$0.9999999\ldots = 1$

$\frac{1}{3}=0.333333\ldots$

Eg. $g(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$

For $x \neq 0$, $-1 \leq \sin \frac{1}{x} \leq 1$

so $-x^2 \leq \underbrace{x^2 \sin \frac{1}{x}}_{g(x)} \leq x^2$, and $-x^2 \to 0$, $g(x) \to 0$, $x^2 \to 0$.

Why is g continuous at 0?

$\lim_{x \to 0} g(x) = 0$ because $-x^2 \leq g(x) \leq x^2$

where $\lim_{x \to 0} (-x^2) = 0 = \lim_{x \to 0} x^2$ so we can use the Squeeze Theorem.

Since $\lim_{x \to 0} g(x) = 0 = g(0)$, g is continuous at 0.

Is g differentiable at 0?

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

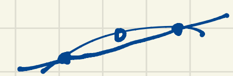
$$= \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} h \sin \frac{1}{h} = 0$$

by the Squeeze Theorem since $-|h| \leq h \sin \frac{1}{h} \leq |h|$ whenever $h \neq 0$

where $\lim_{h \to 0} |h| = 0 = \lim_{h \to 0} (-|h|)$.

$\lim_{h \to 0} \frac{g(a+h) - g(a-h)}{2h}$? slope of secant line between $(a-h, g(a-h))$ and $(a+h, g(a+h))$



$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

$-1 \leq \sin \frac{1}{h} \leq 1$

$-|h| \leq h \sin \frac{1}{h} \leq |h|$

$$g(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

$$g'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Uses the chain rule and the product rule

If $x \neq 0$, $g'(x) = x^2 \left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + 2x \sin\frac{1}{x} = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$

Note: g is differentiable (i.e. everywhere on its domain which is $\mathbb{R}$).

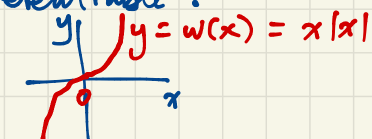
It is not possible to evaluate $g'(0)$ using the chain rule, product rule, rules for derivatives of power functions and trig functions, etc.

$\lim_{x\to 0} g'(x)$ does not exist. So g' is not continuous at 0.

g is differentiable but not continuously differentiable.

Example of a function which is differentiable but not twice differentiable:

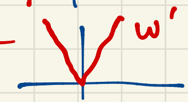
$g(x)$ as above; also $w(x) = x|x| = \begin{cases} x^2, & \text{if } x \geq 0; \\ -x^2, & \text{if } x < 0. \end{cases}$



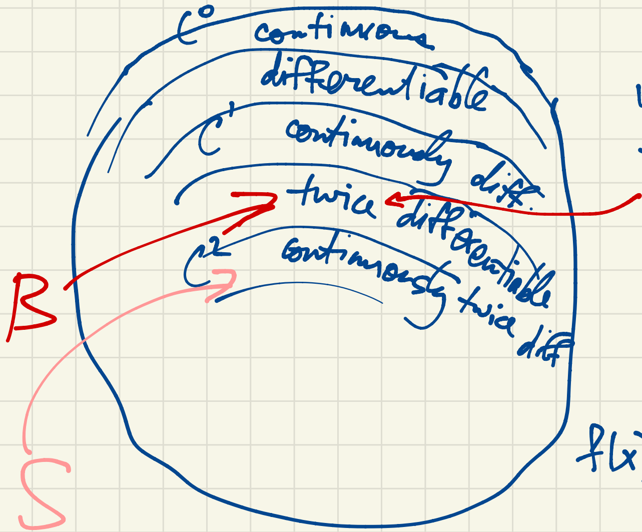
Check: $w'(x) = 2|x|$

$w''(0)$ does not exist.

w is continuously differentiable but not twice differentiable.



Hierarchy of Smoothness:



$C^n = C^n(\mathbb{R})$ is the set of functions $\mathbb{R} \to \mathbb{R}$ which are continuously n times differentiable.

What is an example of a function that is twice differentiable but not continuously twice differentiable? Find $f: \mathbb{R} \to \mathbb{R}$ such that f'' is defined everywhere but f'' is not continuous.

$$f(x) = \begin{cases} x(\ln|x| - 1), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x(\ln|x| - 1) = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x(\ln x - 1) = \lim_{x \to 0^+} \frac{\ln x - 1}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x \to 0^+} (-x) = 0$$

f is now continuous; is it differentiable?

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \frac{h(\ln|h| - 1) - 0}{h} = \lim_{h\to 0} (\ln|h| - 1) = -\infty$$

so this function is not differentiable.

$V(x) = \sin \frac{1}{x^2+1}$ is C^∞ i.e. $V \in C^\infty(\mathbb{R})$ (this function is infinitely differentiable on all of $\mathbb{R}$)

$$V'(x) = \cos \frac{1}{x^2+1} \cdot \left(\frac{(x^2+1)\cdot 0 - 1 \cdot 2x}{(x^2+1)^2} \right) = \frac{-2x}{(1+x^2)^2} \cos \frac{1}{1+x^2}$$

If we continue taking higher and higher order derivatives, every $V^{(n)}(x)$ has terms of the form $\frac{\text{polynomial in } x}{(x^2+1)^k} \cdot \sin\left(\frac{1}{x^2+1}\right)$

$\uparrow$ or $\cos$

$$R(x) = \begin{cases} x^3 \sin \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

If $x \neq 0$, $R'(x) = 3x^2 \sin\frac{1}{x} + x^3 \left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = 3x^2 \sin\frac{1}{x} - x\cos\frac{1}{x}$.

$$R'(0) = \lim_{h \to 0} \frac{R(h) - R(0)}{h} = \lim_{h \to 0} \frac{h^3 \sin\frac{1}{h} - 0}{h} = \lim_{h \to 0} h^2 \sin\frac{1}{h} = 0$$

(see Tuesday's class)

$$R'(x) = \begin{cases} 3x^2 \sin\frac{1}{x} - x\cos\frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

$$R''(x) = 6x \sin\frac{1}{x} + 3x^2 \cdot \cos\frac{1}{x} \cdot \left(\frac{-1}{x^2}\right) - \cos\frac{1}{x} + x \sin\frac{1}{x} \cdot \left(\frac{-1}{x^2}\right)$$

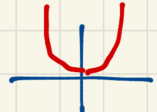
$$= 6x \sin\frac{1}{x} - 4\cos\frac{1}{x} - \frac{1}{x} \sin\frac{1}{x}, \quad \text{if } x \neq 0.$$

$$R''(0) = \lim_{h \to 0} \frac{R'(h) - R'(0)}{h} = \lim_{h \to 0} \frac{3h^2 \sin\frac{1}{h} - h\cos\frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} \left(3h \sin\frac{1}{h} - \cos\frac{1}{h}\right) \text{ does not exist.}$$

So R is not twice differentiable.

Try $S(x) = x^2|x|$



$$S'(x) = \lim_{h\to 0} \frac{S(x+h) - S(x)}{h}$$

If $x>0$, $S(x)=x^3$, $S'(x) = 3x^2$, $S''(x) = 6x$.

If $x<0$, $S(x) = -x^3$, $S'(x) = -3x^2$, $S''(x) = -6x$.

$$S'(0) = \lim_{h\to 0} \frac{h^2|h| - 0}{h} = \lim_{h\to 0} h|h| = 0.$$

Note that $S'(x) = 3x|x|$ for all x. So S' is continuous i.e. S is continuously differentiable ($S \in C^1(\mathbb{R})$).

$$S''(0) = \lim_{h\to 0} \frac{S'(h) - S'(0)}{h} = \lim_{h\to 0} \frac{3h|h| - 0}{h} = \lim_{h\to 0} 3|h| = 0$$

Note: $S''(x) = 6|x|$ for all x

Try $B(x) = \begin{cases} x^4 \sin\frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0 \end{cases}$

$B''(x)$ is defined for all x but not continuous at 0. since $\lim\limits_{x\to0} B''(x)$ does not exist.

For $x \neq 0$, $B'(x) = 4x^3 \sin\frac{1}{x} + x^4 \cdot \left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = 4x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x}$

$$B''(x) = 12x^2 \sin\frac{1}{x} + 4x^3 \cos\frac{1}{x}\left(-\frac{1}{x^2}\right) - 2x\cos\frac{1}{x} - x^2\left(-\sin\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$$

$$= (12x^2 - 1)\sin\frac{1}{x} - 6x\cos\frac{1}{x}$$

$$B'(0) = \lim_{h\to0} \frac{B(h) - B(0)}{h} = \lim_{h\to0} \frac{h^4 \sin\frac{1}{h} - 0}{h} = \lim_{h\to0} h^3 \sin\frac{1}{h} = 0$$

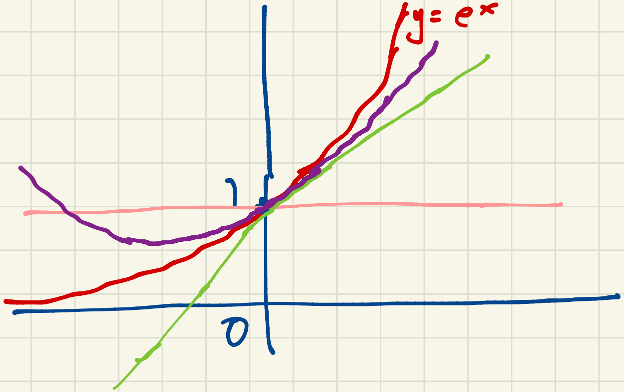
Note: $\lim\limits_{x\to0} B'(x) = \lim\limits_{x\to0} \left(4x^3 \sin\frac{1}{x} - x^2\cos\frac{1}{x}\right) = 0 = B'(0)$ so B' is continuous

i.e. $B \in C^1(\mathbb{R})$ i.e. B is continuously differentiable

$$B''(0) = \lim_{h\to0} \frac{B'(h) - B'(0)}{h}$$

$$= \lim_{h\to0} \frac{4h^3 \sin\frac{1}{h} - h^2\cos\frac{1}{h} - 0}{h} = \lim_{h\to0} \left(4h^2 \sin\frac{1}{h} - h\cos\frac{1}{h}\right) = 0$$

$\lim\limits_{x\to0} B''(x) = \lim\limits_{x\to0} \left((\cancel{12x^2} - 1)\sin\frac{1}{x} - 6x\cos\frac{1}{x}\right)$ does not exist!



Crude (0^{th} order approximation):

$e^x \approx 1$ for $x \approx 0$.

Tangent line approximation: (1^{st} order approximation)

$e^x \approx 1+x$ for $x \approx 0$.

Quadratic approximation (2^{nd} order approximation):

$e^x \approx 1 + x + \frac{x^2}{2}$ for $x \approx 0$

Taylor polynomial $T_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$

is the "best" polynomial approximation of degree n to the function e^x

For a general function $f \in C^n$, the Taylor polynomial of degree n is

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$n! = 1 \times 2 \times 3 \times \cdots \times n$.

Eg. $f(x) = \frac{x}{1+2x}$

$f(0)=0$

$f'(0)=1$

$f''(0)=-4$

$f'''(0)=24$

etc.

$$f'(x) = \frac{(1+2x)\cdot 1 - x\cdot 2}{(1+2x)^2} = \frac{1}{(1+2x)^2}$$

$$f''(x) = \frac{\cancel{(1+2x)^2\cdot 0} - 1\cdot 2(1+2x)\cdot 2}{(1+2x)^4} = \frac{-4}{(1+2x)^3}$$

$$f'''(x) = \frac{\cancel{(1+2x)^3\cdot 0} - (-4)\cdot 3(1+2x)^2\cdot 2}{(1+2x)^6} = \frac{24}{(1+2x)^4}$$

The Taylor series for f centred at 0 is

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\cdot x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \cdots$$

$$= 0 + x - \frac{4}{2}x^2 + \frac{24}{6}x^3 + \cdots$$

$$= x - 2x^2 + 4x^3 - 8x^4 + 16x^5 - 32x^6 + \cdots$$

Note: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$ (geometric series)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$ (geometric series) This converges for $|x|<1$.

$$\frac{1}{1+2x} = 1 - 2x + 4x^2 - 8x^3 + 16x^4 - 32x^5 + \cdots$$ (converges for $|-2x| < 1$ i.e. $|x| < \frac{1}{2}$)

$$\frac{x}{1+2x} = x - 2x^2 + 4x^3 - 8x^4 + 16x^5 - 32x^6 + \cdots$$ (converges for $|x|<\frac{1}{2}$)

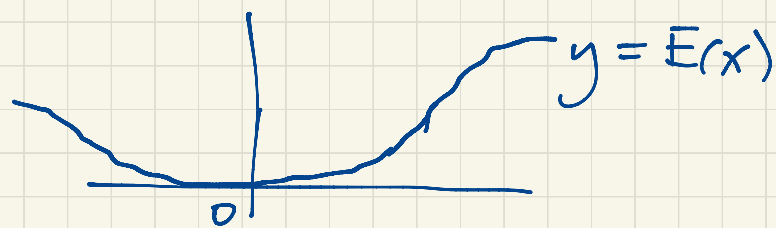
By the way, where does this series converge? (Review from Calc II)

Ratio Test.

Another example: $x^2 \sin x = x^2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)$

$$= x^3 - \frac{x^5}{6} + \frac{x^7}{120} - \frac{x^9}{5040} + \cdots$$ for all x.

Eg. $E(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$



$$E(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases} \qquad \frac{d}{dx}\left(-\frac{1}{x^2}\right) = \frac{d}{dx}\left(-x^{-2}\right) = 2x^{-3}$$

If $x \neq 0$, $E'(x) = e^{-1/x^2}\left(2x^{-3}\right) = 2x^{-3} e^{-1/x^2} = 2\dfrac{e^{-1/x^2}}{x^3}$

$$E''(x) = 2\left(\frac{x^3 \cdot 2e^{-1/x^2}/x^3 - e^{-1/x^2} \cdot 3x^2}{x^6}\right)$$

$$= \frac{(4-6x^2)\, e^{-1/x^2}}{x^6}$$

$$E'''(x) = \frac{x^6 \cdot \left[(-12x)e^{-1/x^2} + (4-6x^2)\left(2e^{-1/x^2}/x^3\right)\right] - (4-6x^2)e^{-1/x^2} \cdot 6x^5}{x^{12}}$$

$$= \frac{x^6(-12x) + (4-6x^2)\, x^3 \cdot 2 - (4-6x^2) \cdot 6x^5}{x^{12}}\, e^{-1/x^2}$$

$$= \frac{x^3(-12x) - (4-6x^2) \cdot 2 - (4-6x^2) \cdot 6x^2}{x^9}\, e^{-1/x^2}$$

$$E^{(n)}(x) = \frac{f_n(x)}{x^{3n}} e^{-1/x^2} \quad \text{when } x \neq 0, \quad \text{for some } f_n(x) \in \mathbb{R}[x].$$

We have seen this by computer for $n = 0, 1, 2, 3$.

How do we know the derivatives of E all have this form?

<u>Proof</u> For each $n \geq 0$ we have a statement about $E^{(n)}(x)$ having a given form.

Since $E(x) = e^{-1/x^2}$, the 0^{th} derivative $E^{(0)}(x) = E(x)$ has the required form with $f_0(x) = 1$.

Now assuming $E^{(n)}(x) = \frac{f_n(x)}{x^{3n}} e^{-1/x^2} = \frac{f_n(x) e^{-1/x^2}}{x^{3n}}$, then

$$E^{(n+1)}(x) = \frac{x^{3n} \cdot \left(f_n'(x) e^{-1/x^2} + f_n(x) \cdot 2e^{-1/x^2}/x^3 \right) - f_n(x) e^{-1/x^2} \cdot 3n\, x^{3n-1}}{x^{6n}}$$