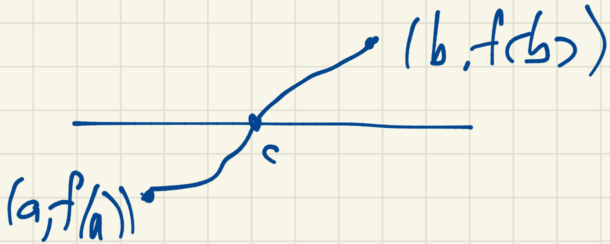
Analysis I (Math 3205)
Fall 2020

Book I

Intermediate Value Theorem

If $f : [a,b] \to \mathbb{R}$ is continuous with $f(a) < 0 < f(b)$, then there exists $c \in (a,b)$ such that $f(c) = 0$.

How does anyone prove this?
what is *your* experience with reading / writing proofs?



The theorem does not hold over $\mathbb{Q}$.

eg. $f(x) = x^2 - 2$, $f :$ {rationals between 0 and 2} $\longrightarrow \mathbb{Q}$ is continuous, $f(0) < 0 < f(2)$ but there is no solution of $f(c) = 0$ in $\mathbb{Q}$.

$\mathbb{Q}$ is not complete; $\mathbb{R}$ is complete.

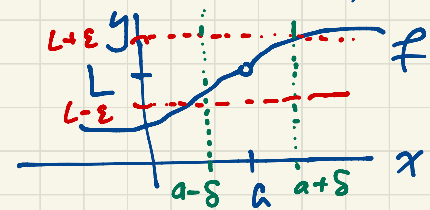
The complete statement of the Intermediate Value Theorem: For all $f : [a,b] \to \mathbb{R}$, if f is continuous and $f(a) < 0 < f(b)$, then there exists $c \in (a,b)$ such that $f(c) = 0$.

"for all", "for every", "for each" : universal quantifiers

"there is", "there exists": existential quantifiers.

For all x there exists y such that $x < y$. (True in $\mathbb{R}$)

There exists y such that for all x, $x < y$. (False in $\mathbb{R}$)



Definition of a Limit

We say $\lim_{x \to a} f(x) = L$ if the following condition holds:

For all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Note: It doesn't matter here what $f(a)$ is or even whether or not it's defined

$f(x)$ is within ε of L

x is within δ of a

i.e. for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all x, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Let's prove that $\lim_{x\to 2}(5x+1) = 11$.

Rough version: $f(x) = 5x+1$. If we need $f(x)$ to be within ε of 11, how close does x have to be to 2 ?

$$|f(x) - 11| < \varepsilon \iff 11-\varepsilon < f(x) < 11+\varepsilon$$
$$\iff 11-\varepsilon < 5x+1 < 11+\varepsilon$$
$$\iff 10-\varepsilon < 5x < 10+\varepsilon$$
$$\iff 2-\frac{\varepsilon}{5} < x < 2+\frac{\varepsilon}{5}$$
$$\iff |x-2| < \frac{\varepsilon}{5}$$

Proof (Actually): Let $\varepsilon > 0$. Then whenever $0 < |x-2| < \frac{\varepsilon}{5}$ we have $2-\frac{\varepsilon}{5} < x < 2+\frac{\varepsilon}{5}$ so

$11-\varepsilon < 5x+1 < 11+\varepsilon$ i.e. $|f(x) - 11| < \varepsilon$.

Another proof: Suppose $\lim_{x\to 7} f(x) = 4$ and $\lim_{x\to 7} g(x) = 5$. Prove that $\lim_{x\to 7}(f(x)+g(x)) = 9$.

Rough version: Given $\varepsilon > 0$ we must find $\delta > 0$ such that $|f(x)+g(x) - 9| < \varepsilon$ whenever $0 < |x-7| < \delta$. Since $\lim_{x\to 7} f(x) = 4$, we can find $\delta > 0$ such that $|f(x) - 4| < \varepsilon$ whenever $0 < |x-7| < \delta$. Also since $\lim_{x\to 7} g(x) = 5$, we can find δ' such that

$|g(x) - 7| < \varepsilon$ whenever $0 < |x-7| < \delta'$.

$4-\varepsilon < f(x) < 4+\varepsilon$ whenever $7-\delta < x < 7+\delta$

$4-\varepsilon < f(x) < 4+\varepsilon$ whenever $7-\delta < x < 7+\delta$ i.e. $|x-7| < \delta$

$5-\varepsilon < g(x) < 5+\varepsilon$ whenever $7-\delta' < x < 7+\delta'$ i.e. $|x-7| < \delta'$

$$\underbrace{9-2\varepsilon < f(x)+g(x) < 9+2\varepsilon} \text{ whenever } 0<|x-7| < \min\{\delta, \delta'\}$$

$$|f(x)+g(x) - 9| < 2\varepsilon \quad \text{whenever} \quad 0 < |x-7| < \min\{\delta, \delta'\}.$$

Actual (final) proof:

Let $\varepsilon > 0$. There exists δ such that $|f(x)-4| < \frac{\varepsilon}{2}$ whenever $0 < |x-7| < \delta$. Also there exists $\delta' > 0$ such that $|g(x)-5| < \frac{\varepsilon}{2}$ whenever $0 < |x-7| < \delta'$. Then

$$|f(x)+g(x)-9| \le |f(x)-4| + |g(x)-5| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $0 < |x-7| < \min\{\delta, \delta'\}$. $\square$

Note: The triangle inequality says $|a+b| \le |a| + |b|$ for all a, b.

$$|f(x)+g(x) - 9| = |f(x) - 4 + g(x) - 5|$$

$$|f(x)-4+g(x)-5| \le |f(x)-4| + |g(x)-5| \qquad |f(x)+g(x)-9| < \varepsilon$$

$$|f(x)+g(x)-9| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$7-2=5 \iff 7-2=\boxed{5}$

$\iff 7=\boxed{5}+2$

$$\left.\begin{array}{l} \infty+1=\infty \iff \infty-\infty=1 \\ \infty+2=\infty \iff \infty-\infty=2 \end{array}\right\} \Rightarrow 1=2.$$

$\lim\limits_{x\to\infty} f(x)=\infty$ <u>means</u> for all M there exist N such that $f(x)>M$ whenever $x>N$.

$[N,\infty)=\{a\in\mathbb{R} : a\geq N\}$.

$(N,\infty)=\{a\in\mathbb{R} : a>N\}$.

$0.9999999\ldots = 1$

$\frac{1}{3}=0.333333\ldots$

Eg. $g(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$

For $x \neq 0$, $\quad -1 \leq \sin\frac{1}{x} \leq 1$

so $\quad -x^2 \leq \underbrace{x^2 \sin\frac{1}{x}}_{g(x)} \leq x^2$

$$-x^2 \to 0, \quad g(x) \to 0, \quad x^2 \to 0$$

Why is g continuous at 0?

$\lim\limits_{x \to 0} g(x) = 0$ because $-x^2 \leq g(x) \leq x^2$

where $\lim\limits_{x \to 0} (-x^2) = 0 = \lim\limits_{x \to 0} x^2$ so we can use the Squeeze Theorem.

Since $\lim\limits_{x \to 0} g(x) = 0 = g(0)$, g is continuous at 0.

Is g differentiable at 0?

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

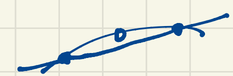
$$= \lim_{h \to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} h \sin\frac{1}{h} = 0$$

by the Squeeze Theorem since $-|h| \leq h \sin\frac{1}{h} \leq |h|$ whenever $h \neq 0$

where $\lim\limits_{h \to 0} |h| = 0 = \lim\limits_{h \to 0} (-|h|)$.

$\lim\limits_{h \to 0} \dfrac{g(a+h) - g(a-h)}{2h}$? slope of secant line between $(a-h, g(a-h))$ and $(a+h, g(a+h))$



$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

$$-1 \leq \sin\frac{1}{h} \leq 1$$

$$-|h| \leq h \sin\frac{1}{h} \leq |h|$$

$$g(x) = \begin{cases} x^2 \sin\frac{1}{x} &, \text{ if } x \neq 0; \\ 0 &, \text{ if } x = 0. \end{cases}$$

$$g'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x} &, \text{ if } x \neq 0; \\ 0 &, \text{ if } x = 0. \end{cases}$$

Uses the chain rule and the product rule

If $x \neq 0$, $g'(x) = x^2 \left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + 2x \sin\frac{1}{x} = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$

Note: g is differentiable (i.e. everywhere on its domain which is $\mathbb{R}$).

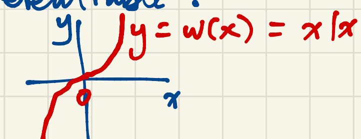
It is not possible to evaluate $g'(0)$ using the chain rule, product rule, rules for derivatives of power functions and trig functions, etc.

$\lim_{x \to 0} g'(x)$ does not exist. So g' is not continuous at 0.

g is differentiable but not continuously differentiable.

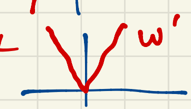
Example of a function which is differentiable but not twice differentiable:

$g(x)$ as above; also $W(x) = x|x| = \begin{cases} x^2, & \text{if } x \geq 0; \\ -x^2, & \text{if } x < 0. \end{cases}$

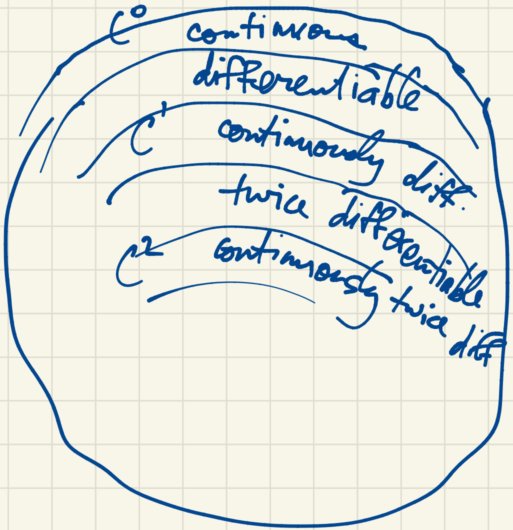


Check: $w'(x) = 2|x|$

$w''(0)$ does not exist. w is continuously differentiable but not twice differentiable.



Hierarchy of Smoothness:



$C^n = C^n(\mathbb{R})$ is the set of functions $\mathbb{R} \to \mathbb{R}$ which are continuously n times differentiable.