Analysis I (Math 3205)
Fall 2020

Book 3

Let $(a_n)_n$ be a sequence of real numbers. It is possible for such a sequence to have no limit point e.g. $a_n = n$. The sequence of positive integers has only isolated points. However, if (a_n) is bounded then it must have at least one limit point by the Bolzano-Weierstrass Theorem.

Eg. consider the sequence $(\sin n)_{n\in\mathbb{N}} = (\sin 1, \sin 2, \sin 3, \sin 4, \dots)$.

This sequence diverges. But the sequence is bounded (all terms lie in $[-1,1]$). So the sequence has a convergent subsequence. Thus there is at least one limit point. All limit points must lie in $[-1,1]$.

$\sin 0 = 0.000\dots$

$\sin 1 = 0.841\dots$

$\sin 2 = 0.909\dots$

$\vdots$

$\sin 22 = -0.009$

$\sin 44 = 0.018$

$\sin 45 = 0.850$

$\sin 46 = 0.902$

$\pi \approx \frac{22}{7}$

$7\pi \approx 22$

$\sin 22 \approx \sin 7\pi = 0$

$\sin n \neq 0$ for any positive integer n because $\pi \notin \mathbb{Q}$.

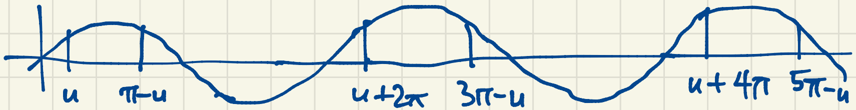
$\sin x = 0 \iff x = k\pi$ for some $k \in \mathbb{Z}$

Also since $\pi \notin \mathbb{Q}$, the sequence $(\sin n)_n$ has no repeated terms and the limit points of $(\sin n)_n$ are all points of $[-1,1]$.

$\left(\pi = \frac{n}{k} \in \mathbb{Q} \right. \Longleftarrow \sin n = 0 \iff n = k\pi$ for some $k \in \mathbb{Z}$

If π is irrational then the sequence $(\sin n)_n$ has distinct terms (it never repeats).

Why? If $\sin u = \sin v$ then either $v-u = 2k\pi$ for some $k\in\mathbb{Z}$
or $v+u = (2k+1)\pi$ for some $k\in\mathbb{Z}$.



So if $\sin m = \sin n$ where $m\neq n$ are integers then either $m-n = 2k\pi$ with $0\neq k\in\mathbb{Z}$ so $\pi = \frac{m-n}{2k}\in\mathbb{Q}$; or $m+n = (2k+1)\pi$ for some $k\in\mathbb{Z}$ so $\pi = \frac{m+n}{2k+1}\in\mathbb{Q}$ again contradicting $\pi\notin\mathbb{Q}$.

$0! = 1$
$1! = 1$
$2! = 1\times 2 = 2$
$3! = 1\times2\times3 = 6$
$4! = 1\times2\times3\times4 = 24$

Let's prove $\pi\notin\mathbb{Q}$. Warm-up: prove $e\notin\mathbb{Q}$.

$$e = \sum_{n=0}^{\infty}\frac{1}{n!} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Recall: $$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

$$e = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots$$

Suppose $e\in\mathbb{Q}$; say $e = \frac{a}{b}$ in lowest terms ($a,b\in\mathbb{N}$, $\gcd(a,b)=1$).

$$\frac{a}{b} = e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

Suppose $e \in \mathbb{Q}$; say $e = \frac{a}{b}$ in lowest terms ($a, b \in \mathbb{N}$, $\gcd(a,b) = 1$).
Multiply both sides by $b! = 1 \times 2 \times 3 \times \cdots \times (b-1)b$.

$$b!\,\frac{a}{b} = \overbrace{(b-1)!\,a}^{\text{integer}} = b!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(b-1)!} + \frac{1}{b!} + \frac{1}{(b+1)!} + \cdots\right)$$

$$= \underbrace{b! + b! + \frac{b!}{2!} + \frac{b!}{3!} + \cdots + \underset{\underset{b}{\parallel}}{\frac{b!}{(b-1)!}} + \underset{\underset{1}{\parallel}}{\frac{b!}{b!}}}_{\text{integers}} + \underbrace{\underset{\underset{\frac{1}{b+1}}{\parallel}}{\frac{b!}{(b+1)!}} + \underset{\underset{\frac{1}{(b+1)(b+2)}}{\parallel}}{\frac{b!}{(b+2)!}}}_{\text{Not an integer by comparison test}} + \cdots$$

The series $\frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \cdots$ converges by comparison with

$$\frac{1}{b+1} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \frac{1}{(b+1)^4} + \cdots = \frac{1/(b+1)}{1 - \frac{1}{b+1}} = \frac{1}{(b+1)-1} = \frac{1}{b} < 1.$$

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r} \quad (\text{for } |r| < 1).$$

This is a contradiction. So $e \notin \mathbb{Q}$.

$$(uv)' = u'v + uv'$$

$$(uv)'' = (u'v+uv')' = u''v + u'v' + u'v' + uv'' = u''v + 2u'v' + uv''$$

$$(uv)''' = (u''v+2u'v'+uv'')' = (u'''v + u''v') + 2(u''v'+u'v'') + (u'v''+uv''')$$

$$= u'''v + 3u''v' + 3u'v'' + uv'''$$

$$(u+v)^1 = u+v$$

$$(u+v)^2 = u^2 + 2uv + v^2$$

$$(u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$$

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$$(u+v)^n = \sum_{k=0}^{n} \binom{n}{k} u^k v^{n-k}$$

(Binomial Theorem)

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \in \mathbb{Z}$$

"binomial coefficients" are the entries in Pascal's Triangle

Leibniz' Formula

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}$$

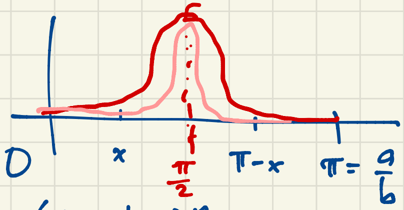
Theorem $\pi \notin \mathbb{Q}$.

Proof Suppose $\pi = \frac{a}{b}$ in lowest terms (i.e. $a, b \in \mathbb{N}$, $\gcd(a,b)=1$). We look for a contradiction. Consider the function $f(x) = \frac{1}{n!} x^n (a-bx)^n$ where $n \in \mathbb{N}$ will be chosen later. Note: $f(x) = u(x)v(x)$ where $u(x) = \frac{1}{n!} x^n$, $v(x) = (a-bx)^n$.

Lemma For every $k \geq 0$, $f^{(k)}(0) = (-1)^k f^{(k)}(\pi) \in \mathbb{Z}$.

Proof $f(\pi - x) = f(\frac{a}{b} - x) = \frac{1}{n!}(\frac{a}{b} - x)^n (a - b(\frac{a}{b} - x))^n = \frac{1}{n!}(\frac{a}{b} - x)^n (a - (a - bx))^n$

$= \frac{1}{n!}(\frac{a}{b} - x)^n (bx)^n = \frac{1}{n!}(\frac{a}{b} - x)^n b^n x^n = \frac{1}{n!}((\frac{a}{b} - x)b)^n x^n = \frac{1}{n!}(a-bx)^n x^n = f(x)$.



$f(x) = \frac{1}{n!}(ax - bx^2)^n$

$$f(\pi - x) = f(x)$$
$$-f'(\pi - x) = f'(x)$$
$$f''(\pi - x) = f''(x)$$
$$-f'''(\pi - x) = f'''(x)$$

$$f^{(k)}(x) = (-1)^k f^{(k)}(\pi - x)$$
$$f^{(k)}(0) = (-1)^k f^{(k)}(\pi)$$

We must show this $\in \mathbb{Z}$.

$f(x) = u(x)v(x)$,

$$u^{(k)}(0) = \begin{cases} 0 & \text{if } k \neq n; \\ 1 & \text{if } k = n. \end{cases}$$

$\in \mathbb{Z}$

$$u(x) = \frac{1}{n!} x^n$$
$$u'(x) = \frac{1}{n!} \cdot n x^{n-1} = \frac{1}{(n-1)!} x^{n-1}$$
$$u''(x) = \frac{1}{(n-2)!} x^{n-2}$$

$\cdots$

$$u^{(n-2)}(x) = \frac{1}{2!} x^2 = \frac{1}{2} x^2$$
$$u^{(n-1)}(x) = x$$
$$u^{(n)}(x) = 1$$
$$u^{(n+1)}(x) = 0$$
$$u^{(n+2)}(x) = 0 \text{ etc.}$$

Recall: $f(x) = u(x)v(x)$, $u(x) = \frac{1}{n!}x^n$, $v(x) = \underbrace{(a-bx)^n}_{}\in \mathbb{Z}[x]$

i.e. a polynomial in x with integer coefficients

$$f^{(k)}(x) = \sum_{r=0}^{k} \binom{k}{r} u^{(r)}(x)\, v^{(k-r)}(x)$$

$$f^{(k)}(0) = \sum_{r=0}^{k} \underbrace{\binom{k}{r}}_{\text{integers}} \underbrace{u^{(r)}(0)}_{\text{integers}}\, \underbrace{v^{(k-r)}(0)}_{\text{integers}} \in \mathbb{Z}.$$

This proves the Lemma.

Note: $f(x)$ is a poly. in x of degree $2n$.

Return to the Theorem.

$$F''(x) = \cancel{f''(x) - f^{(4)}(x)} + \cancel{f^{(6)}(x)} - \cdots + \cancel{(-1)^{n-1} f^{(2n)}(x)}$$

Consider $F(x) = f(x) - \cancel{f''(x)} + \cancel{f^{(4)}(x)} - \cancel{f^{(6)}(x)} + \cdots + \cancel{(-1)^n f^{(2n)}(x)}$

$$\frac{d}{dx}\left[F'(x)\sin x - F(x)\cos x\right] = F''(x)\sin x + \cancel{F'(x)\cos x} - \left(\cancel{F'(x)\cos x} - F(x)\sin x\right)$$

$$= \left[F''(x) + F(x)\right]\sin x = f(x)\sin x$$

$$\int_0^{\pi} f(x)\sin x\, dx = \left[\cancel{F'(x)\sin x} - F(x)\cos x\right]_0^{\pi} = F(\pi) - F(0) = F(0) - F\left(\tfrac{a}{b}\right) \in \mathbb{Z}$$ by the Lemma

$$0 < \int_0^{\pi} f(x)\sin x\, dx < \pi \cdot f\left(\tfrac{\pi}{2}\right) = \frac{\pi}{n!}\left(\frac{\pi}{2}\right)^{2n} \to 0 \text{ as } n\to\infty.$$

$f(x) = \frac{1}{n!}(ax-bx^2)^n$ is maximized at $x = \frac{\pi}{2} = \frac{a}{2b}$ on $[0,\pi]$. For n sufficiently large the integral is in $(0,1)$, it can't be an integer □