Analysis I (Math 3205)
Fall 2020

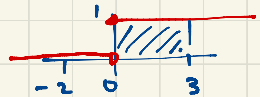
Book 2

$$\left\{\begin{array}{c}\text{Lebesgue}\\ \text{integrable}\\ \text{functions}\end{array}\right\} \supset \left\{\begin{array}{c}\text{Riemann}\\ \text{integrable}\\ \text{functions}\end{array}\right\} \supset \left\{\begin{array}{c}\text{continuous}\\ \text{functions}\end{array}\right\} \supset \left\{\begin{array}{c}\text{differentiable}\\ \text{functions}\end{array}\right\} \supset \left\{\begin{array}{c}\text{continuously}\\ \text{differentiable}\\ \text{functions}\end{array}\right\} \supset \cdots$$

$\Downarrow$ g Dirichlet function

$\Downarrow$ H Heaviside function

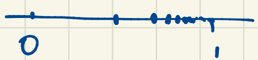
$$H(x) = \begin{cases} 1 & \text{if } x \geq 0; \\ 0 & \text{if } x < 0 \end{cases}$$



$$\int_{-2}^{3} H(x)\,dx = 3$$

$S = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots\}$ is bounded: 0 is a lower bound, 1 is an upper bound.

$\frac{1}{2}$ is the greatest lower bound.



Every $m \leq \frac{1}{2}$ is a lower bound for S, meaning $s \geq \frac{1}{2}$ for all $s \in S$.

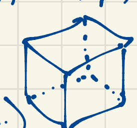
(so $\frac{1}{2}$ is a lower bound) and it is the greatest lower bound.

1 is the least upper bound of S.

Fact: $|[0,1]| = |\mathbb{R}^3|$

Basic idea of the proof: $|(0,1)| = |(0,1)^3|$

$(0,1)^3 = (0,1) \times (0,1) \times (0,1) = \{(x,y,z) : 0 < x,y,z < 1\}$



Bijection: $a \longmapsto (0.a_1 a_4 a_7 a_{10} a_{13} \cdots, 0.a_2 a_5 a_8 a_{11} a_{14} \cdots, 0.a_3 a_6 a_9 a_{12} a_{15} \cdots)$

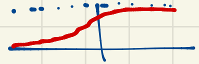
$0 < a < 1$

$\underbrace{0.14159265358\dots}_{\pi - 3} \longmapsto (0.1565\cdots, 0.4958\cdots, 0.1239\cdots)$

$a = 0.a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} \cdots$

$= a_1 \cdot 10^{-1} + a_2 \cdot 10^{-2} + a_3 \cdot 10^{-3} + a_4 \cdot 10^{-4} + \cdots, \quad a_i \in \{0, 1, 2, \cdots, 9\}$

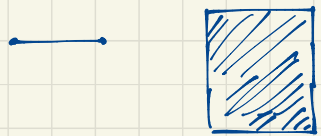
$|(0,1)| = |\mathbb{R}|$



see video on Cardinality

$|(0,1)| = |(0,1)^3| = |\mathbb{R}^3|$ and this bijection can be given constructively i.e. by an explicit formula (in particular this is a theorem in ZF, not requiring the Axiom of Choice)

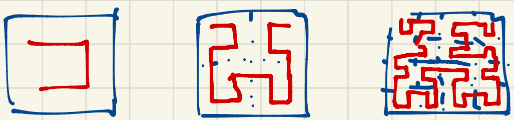
$|[0,1]| = |[0,1]^2|$



There is a bijection $[0,1] \to [0,1]^2$ but no continuous bijection.

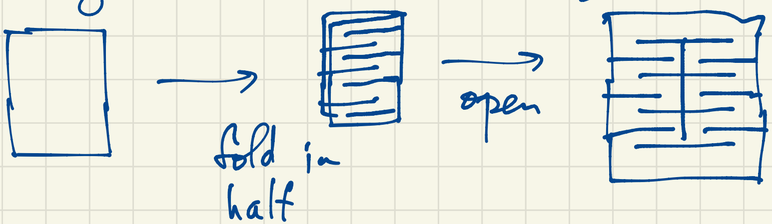
However there is a continuous surjection (map that is onto)

This gives a "space-filling curve": it goes through every point of the square.



etc.

How do you cut a hole in an $8\frac{1}{2}'' \times 11''$ sheet of paper that you can walk through?



Fact: There is a set of open intervals in $\mathbb{R}$ of total length less than 1 which covers all the rational numbers.

Since $\mathbb{Q}$ is countable, $\mathbb{Q} = \{a_1, a_2, a_3, a_4, a_5, \ldots\}$. Then

$$\mathbb{Q} \subseteq \bigcup_{n=1}^{\infty} \left(a_n - \tfrac{1}{2^{n+1}},\ a_n + \tfrac{1}{2^{n+1}}\right) = \left(a_1 - \tfrac{1}{4}, a_1 + \tfrac{1}{4}\right) \cup \left(a_2 - \tfrac{1}{8}, a_2 + \tfrac{1}{8}\right) \cup \left(a_3 - \tfrac{1}{16}, a_3 + \tfrac{1}{16}\right) \cup \left(a_4 - \tfrac{1}{32}, a_4 + \tfrac{1}{32}\right) \cup \cdots$$

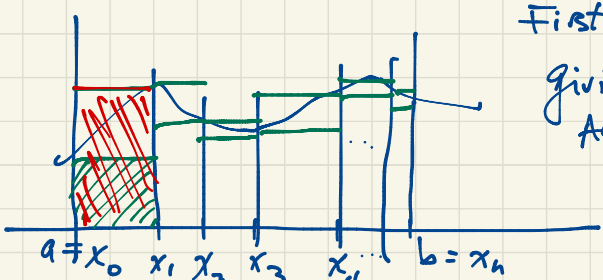
Total length $< \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$

Once again the set of intervals can be given constructively ie. explicitly with no need for the Axiom of Choice.

What is a (Riemann) integral? i.e. the integral as defined in Calculus I-II ?

Suppose $f: [a,b] \to \mathbb{R}$. We want to define $\int_a^b f(x)\,dx$. We start with lower and upper bounds for the integral (these being upper and lower Riemann sums).

We then take sup{lower Riemann sums} and inf{upper Riemann sums}.



First subdivide $[a,b]$ at points $a = x_0 \le x_1 \le x_2 \le \cdots \le x_n = b$ giving n subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$.

Assuming f is bounded, $m_i \le f(x) \le M_i$ on $[x_{i-1}, x_i]$

where $M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\} = \sup_{[x_{i-1}, x_i]} f$, $\quad m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\} = \inf_{[x_{i-1}, x_i]} f$

The Riemann Sums corresponding to the partition $a = x_0 \le x_1 \le x_2 \le \cdots \le x_{n-1} \le x_n = b$ of $[a,b]$ are:

Upper sum $\sum_{i=1}^{n} \underbrace{(x_i - x_{i-1})}_{\text{base}} \underbrace{M_i}_{\text{height}} = (x_1 - x_0)M_1 + (x_2 - x_1)M_2 + \cdots + (x_n - x_{n-1})M_n$.

Lower sum $\sum_{i=1}^{n} (x_i - x_{i-1}) m_i$

We should have $\underset{\sum (x_i - x_{i-1})m_i}{\text{Lower sum}} \le \int_a^b f(x)\,dx \le \underset{\sum (x_i - x_{i-1})M_i}{\text{Upper sum}}$

We can't just let $n \to \infty$. By the Least Upper Bound Property, sup{lower bounds} exists and inf{upper bounds} exists. And

$$\sup\{\text{lower bounds}\} \le \inf\{\text{upper bounds}\}.$$

If these two values agree, this gives a definite value for $\int_a^b f(x)\,dx$.

For lots of functions (eg. the Heaviside function and for all continuous functions), this works. For Dirichlet's function, the Riemann integral $\int_0^1 g(x)\,dx$ is undefined.

why? For Dirichlet's function $g(x)=\begin{cases}1, & \text{if } x\in\mathbb{Q};\\ 0, & \text{if } x\notin\mathbb{Q}\end{cases}$

$m_i = 0, \quad M_i = 1$ for each $i = 1,2,\dots,n$

Upper sums: $\displaystyle\sum_{i=1}^{n}(x_i-x_{i-1})\underbrace{M_i}_{1} = (\cancel{x_1}-x_0)+(\cancel{x_2}-\cancel{x_1})+(\cancel{x_3}-\cancel{x_2})+\cdots+(\cancel{x_{n-1}}-\cancel{x_{n-2}})+(x_n-\cancel{x_{n-1}})$

$= x_n - x_0 = b-a = 1-0 = 1$

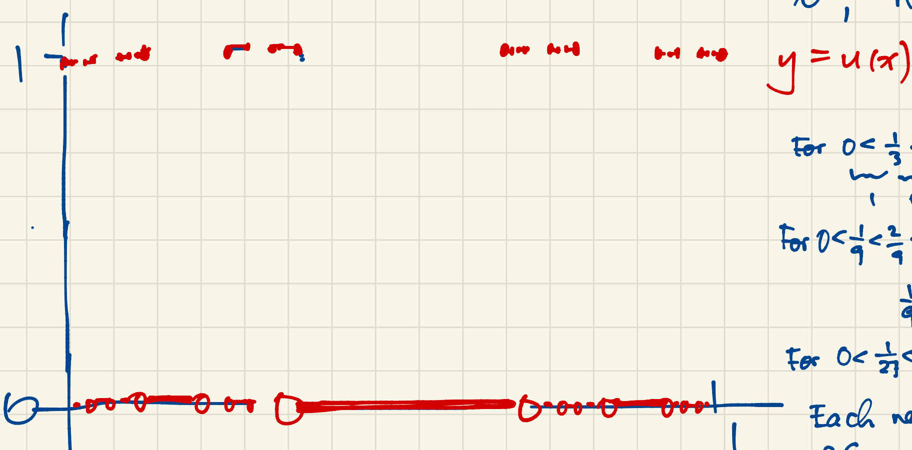
Lower sums: $\displaystyle\sum_{i=1}^{m}(x_i-x_{i-1})\underbrace{m_i}_{0} = 0+0+\cdots+0 = 0.$

For C the Cantor set define $u(x)=\begin{cases}1, & \text{if } x\in C\\ 0, & \text{if } x\notin C.\end{cases}$

What is $\int_0^1 u(x)\,dx$?

Lower sums are 0.

Upper sums can be made as small as we want.



For $0<\underbrace{\tfrac13}_{1}<\underbrace{\tfrac23}_{0}<\underbrace{1}_{1}$ the Riemann Sum is $\frac13\cdot1+\frac13\cdot0+\frac13\cdot1=\frac23$.

For $0<\frac19<\frac29<\frac13<\frac23<\frac79<\frac89<1$ the Riemann Sum is

$\frac19\cdot1+\frac19\cdot0+\frac19\cdot1+\frac13\cdot0+\frac19\cdot1+\frac19\cdot0+\frac19\cdot1=\frac49$

For $0<\frac1{27}<\frac2{27}<\cdots<\frac{26}{27}<1$ the corresponding Riemann Sum is $\frac{8}{27}$.

Each new upper Riemann Sum is $\frac23$ of the previous one ie $\left(\frac23\right)^n$.

$\inf\{\text{upper Sums}\} = 0.$

For the function $u(x)$, $\underbrace{\sup\{\underbrace{\text{lower sums}}_{0}\}}_{0} \leq \int_0^1 u(x)\,dx \leq \underbrace{\inf\{\text{upper sums}\}}_{0}$

So $\int_0^1 u(x)\,dx = 0$.

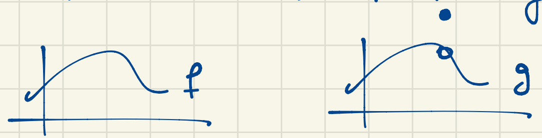
Note: $u(x)$ has infinitely many discontinuities but it is not discontinuous everywhere.
$u(x)$ is continuous on a set of open intervals inside $[0,1]$ of total length 1.
The total length of the Cantor set C (where $u=1$) is 0.
However C is uncountable; $|C| = |\mathbb{R}|$. Why?
Every $a \in [0,1]$ has a ternary expansion

$$a = 0.a_1a_2a_3a_4a_5\cdots \qquad (a_i \in \{0,1,2\})$$
$$= \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \frac{a_4}{3^4} + \frac{a_5}{3^5} + \cdots$$

The points in C are those with $a_i \in \{0,2\}$ only.
$|C| = |\mathbb{R}| = |[0,1]|$. A bijection $C \longrightarrow [0,1]$

$$\underset{\substack{\text{(base 3)}\\ \text{ternary}}}{0.20022202002\cdots} \longmapsto \underset{\substack{\text{(base 2)}\\ \text{binary}}}{0.10011101001\cdots}$$

If two functions f and g agree except at a single point, the $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$

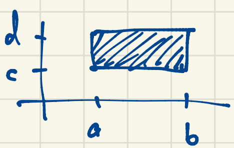


The same holds for changing a function at any finite number of points.

We want to be able to measure sets to distinguish their size, not as cardinality, but length (in one dimension), area (in two dimensions), volume (in 3 dimensions), etc.
Defining measure of a set is equivalent to being able to integrate.
In one dimension, $\lambda([a,b]) = b-a$ for $a \le b$. (the length)

Greek letter lambda)

In two dimensions, $\lambda([a,b] \times [c,d]) = (b-a)(d-c)$



Cartesian product $\{(x,y) : x \in [a,b],\ y \in [c,d]\}$.

Borel measure extends this notion to larger sets and more complicated constructions.
Borel measure extends to Lebesgue measure which is the gold standard for measuring sets.

Lebesgue measure of $A \subseteq \mathbb{R}^n$ is denoted $\lambda(A)$.

$\lambda([a,b]) = b-a$ for $a \le b$

$\lambda(A) \geq 0$ for all measurable A.

$\lambda(\{a\}) = 0$

$\lambda(A \sqcup B) = \lambda(A) + \lambda(B)$

disjoint union.

This extends to countable disjoint unions:

$$\lambda\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \lambda(A_i)$$

If $A \subseteq B$ then $\lambda(A) \le \lambda(B)$.

$\lambda(\mathbb{Q}) = 0$. This follows from the properties above:

$$\mathbb{Q} = \{a_1, a_2, a_3, \dots\} = \bigsqcup_{i=1}^{\infty} \{a_i\}$$

singleton sets (sets with single elements)

$$\Rightarrow \lambda(\mathbb{Q}) = \sum_{i=1}^{\infty} \lambda(\{a_i\})$$

$$= \sum_{i=1}^{\infty} 0 = 0.$$

$\mathbb{R} = \bigsqcup_{a \in \mathbb{R}} \{a\}$ but this is not a countable union

so $\lambda(\mathbb{R}) \neq 0$.

Recall, as observed about 5 slides back,

$$\mathbb{Q} \subset \underbrace{\bigcup_i \left(a_i - \frac{1}{2^{i+1}}, a_i + \frac{1}{2^{i+1}}\right)}_{\text{set of Lebesgue measure } < 1.}$$

<u>Sets of measure zero</u> are sets which can be covered by countable unions of intervals of total length as small as we want (ie. for every $\varepsilon > 0$, the set is covered by intervals of total length $< \varepsilon$). Such sets are considered 'negligible' in the sense of length ie. measure.

Sets of Lebesgue measure zero have $\lambda(A) = 0$.

eg. $\lambda(\mathbb{Q}) = 0$ so $\mathbb{Q}$ has Lebesgue measure zero.

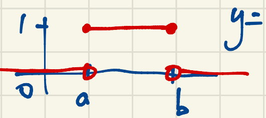
Also the Cantor set $C \subset [0,1]$ has measure zero.

$\mathbb{Q}$ is countable and C is uncountable so from the perspective of cardinality, there is a big difference in size between these two sets. But in terms of length (Lebesgue measure), both have measure zero: $\lambda(\mathbb{Q}) = \lambda(C) = 0$.

Connection between measure and integration:

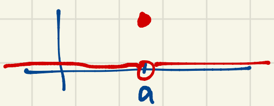
Given a set $A \subseteq \mathbb{R}$, its <u>characteristic function</u> $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

Greek "chi"



$y = \chi_{[a,b]}(x)$

$$\int_{-\infty}^{\infty} \chi_{[a,b]}(x)\,dx = b-a = \lambda([a,b])$$



$$\int_{-\infty}^{\infty} \chi_{\{a\}}(x)\,dx = 0 = \lambda(\{a\})$$

Similarly,

$$\int_{-\infty}^{\infty} \chi_C(x)\,dx = \lambda(C) = 0$$

where $C \subset [0,1]$ is the Cantor set (and this integral is defined as both a Riemann integral and as a Lebesgue integral).

In general, $\displaystyle\int_{-\infty}^{\infty} \chi_A(x)\,dx = \lambda(A)$.

$\chi_{\mathbb{Q}} = g$ is Dirichlet's function

$\displaystyle\int_{-\infty}^{\infty} \chi_{\mathbb{Q}}(x)\,dx = \lambda(\mathbb{Q}) = 0$. (This however is the <u>Lebesgue integral</u>, not the Riemann integral of Calculus I and II). The Riemann integral is undefined.

If f and g agree except at a finite number of points, $\underbrace{\int_a^b f(x)\,dx}_{\int_{[a,b]} f} = \int_a^b g(x)\,dx$

More generally, if f and g agree almost everywhere (i.e. except on a set of measure zero) then $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ for every interval $[a,b]$.

f and g agree almost everywhere (f and g agree a.e.)

$\Longleftrightarrow \lambda(\{x \in \mathbb{R} : f(x) \neq g(x)\}) = 0$

This is an important example of an equivalence relation.

If $f = g$ a.e. and $g = h$ a.e. then $f = h$ a.e.

$f = f$ a.e.

If $f = g$ a.e. then $g = f$ a.e.

$\lambda(A \sqcup B) = \lambda(A) + \lambda(B)$ for all measurable sets A, B.

If B is a closed unit ball in $\mathbb{R}^3$ then $\lambda(B) = \frac{4\pi}{3}$ (volume)

radius 1

$B = A_1 \sqcup \cdots \sqcup A_5$ where $A_1, \cdots, A_5$ can be repositioned to form two unit balls of total Lebesgue measure (volume) $\frac{8\pi}{3}$.

$$\lambda(B) \stackrel{?}{=} \underbrace{\cancel{\lambda(A_1) + \cdots + \lambda(A_5)}}_{\text{undefined}} = 2\lambda(B).$$

$A_1, \cdots, A_5$ are non-measurable.

Sequences $a_1, a_2, a_3, a_4, \dots$

The limit of a sequence $(a_n)_{n\in\mathbb{N}}$ is L if:

For all $\varepsilon > 0$, there exists N such that $|a_n - L| < \varepsilon$ whenever $n > N$.

Eg. the sequence $\left(\frac{n}{2n+1}\right)_{n\in\mathbb{N}} = \left(\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \dots\right)$ converges to $\frac{1}{2}$.

$\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$ i.e. $(a_n) \to \frac{1}{2}$ i.e. $a_n \to \frac{1}{2}$ as $n \to \infty$.

In this case $\lim_{x\to\infty} \frac{x}{2x+1} = \frac{1}{2}$ follows from $\lim_{x\to\infty} \frac{x}{2x+1} = \lim_{x\to\infty} \frac{1}{2+\frac{1}{x}} = \frac{1}{2+0} = \frac{1}{2}$.

Eg. $(\sin n\pi)_{n\in\mathbb{N}} = (0, 0, 0, \dots)$ converges to 0.

$(\sin n\pi)_{n\in\mathbb{N}} \to 0$

But $\lim_{x\to\infty} \sin(\pi x)$ does not exist.

Some sequences are defined recursively eg. the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$

Eg. consider the sequence $a_n = \begin{cases} 0, & \text{for } n=1; \\ \frac{1}{5}(2a_{n-1}+7), & \text{for } n=2,3,4,\dots \end{cases}$

This is a <u>recursive</u> definition.

n	1	2	3	4	5	6
a_n	0	$\frac{7}{5}$	$\frac{49}{25}$	$\frac{273}{125}$	$\frac{1421}{625}$	$\frac{7217}{3125}$

etc.

$$a_1 \xrightarrow{f} a_2 \xrightarrow{f} a_3 \xrightarrow{f} a_4 \xrightarrow{f} a_5 \longrightarrow \dots$$

$$a_{n+1} = f(a_n)$$

where $f(x) = \frac{1}{5}(2x+7)$.

Iteration: repetition.

The n^{th} term of the sequence is $a_n = f^{n-1}(a_1)$ where $f^{n-1} = \underbrace{f \circ f \circ f \circ \dots \circ f}_{n-1}$

$$a_2 = f(a_1)$$
$$a_3 = f(f(a_1))$$
$$a_4 = f(f(f(a_1))) = f^3(a_1)$$
etc.

Let's look at the sequence in decimal approximations to get a better ideal of its behavior.

From our computer session it appears that $(a_n)_n$ is increasing and converges to $2\frac{1}{3} = \frac{7}{3}$.
Let's prove $(a_n)_n \to \frac{7}{3}$. We'll do this in two ways. One way is to find an explicit formula for a_n. It is obvious that a_n has denominator 5^{n-1} but what is its numerator? We don't see an obvious pattern yet.

But first, why is it no surprise that the limit is $\frac{7}{3}$?

If the sequence converges to L then

$$a_{n+1} = \frac{1}{5}(2a_n + 7) \qquad \text{for } n = 1, 2, 3, \ldots$$

where we can take the limit on both sides as $n \to \infty$ and get

$$L = \frac{1}{5}(2L + 7)$$
$$5L = 2L + 7$$
$$3L = 7$$
$$L = \frac{7}{3}.$$

But this doesn't prove $(a_n)_n \to \frac{7}{3}$ since we assumed the sequence converges.

How do we know this?

Look at $a_n - \frac{7}{3}$ (which should converge to zero) and see if this exhibits a pattern.

From the table of values it appears that $\frac{7}{3} - a_n = \frac{7}{3} \cdot \left(\frac{2}{5}\right)^{n-1}$ for $n = 1, 2, 3, \ldots$

i.e. $a_n = \frac{7}{3}\left[1 - \left(\frac{2}{5}\right)^{n-1}\right]$ for $n = 1, 2, 3, \ldots$ which is an explicit formula for a_n.

Let's prove this formula by induction. When $n = 1$, $a_1 = 0$ and this agrees with the formula which gives $\frac{7}{3}\left[1 - \left(\frac{2}{5}\right)^0\right] = 0$.

Assuming $a_n = \frac{7}{3}\left[1-\left(\frac{2}{5}\right)^{n-1}\right]$ for some positive integer n,

$$a_{n+1} = \frac{1}{5}(2a_n+7) = \frac{1}{5}\left(2\times\frac{7}{3}\left[1-\left(\frac{2}{5}\right)^{n-1}\right]+7\right) = \frac{14}{15}\left[1-\left(\frac{2}{5}\right)^{n-1}\right]+\frac{7}{5}$$

$$= \frac{14}{15} - \frac{14}{15}\left(\frac{2}{5}\right)^{n-1} + \frac{21}{15} = \frac{35}{15} - \frac{14}{15}\left(\frac{2}{5}\right)^{n-1} = \frac{7}{3} - \frac{7}{3}\cdot\frac{2}{5}\left(\frac{2}{5}\right)^{n-1} = \frac{7}{3}-\frac{7}{3}\left(\frac{2}{5}\right)^{n}$$

$= \frac{7}{3}\left[1-\left(\frac{2}{5}\right)^{n}\right]$, which is correctly predicted by our explicit formula.

By induction, the conjectured explicit formula for a_n holds for all $n = 1, 2, 3, 4, \dots$. $\square$

At this point, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{7}{3}\left[1-\left(\frac{2}{5}\right)^{n-1}\right] = \frac{7}{3}[1-0] = \frac{7}{3}$.

Alternative proof: Substitute $b_n = \frac{7}{3} - a_n$. This gives a new sequence which also satisfies a recursive formula

$$b_{n+1} = \frac{7}{3} - a_{n+1} = \frac{7}{3} - f(a_n) = \frac{7}{3} - \frac{1}{5}(2a_n+7) = \frac{14}{15} - \frac{2}{5}a_n = \frac{14}{15} - \frac{2}{5}\left(\frac{7}{3}-b_n\right) = \frac{2}{5}b_n.$$

i.e. $b_n = \begin{cases} \frac{7}{3}, & \text{if } n=1; \\ \frac{2}{5}b_{n-1}, & \text{for } n = 2,3,4,\dots \end{cases}$

n	1	2	3	4	5
b_n	$\frac{7}{3}$	$\frac{7}{3}\cdot\frac{2}{5}$	$\frac{7}{3}\cdot\left(\frac{2}{5}\right)^2$	$\frac{7}{3}\left(\frac{2}{5}\right)^3$	$\frac{7}{3}\left(\frac{2}{5}\right)^4$

etc.

$$b_n = \frac{7}{3}\cdot\left(\frac{2}{5}\right)^{n-1}$$

$A = B$ "$\Rightarrow$" "$\longrightarrow$"

implies

can be used in limit notation eg. $(a_n) \to 5$

$\lim_{x \to 3} f(x) = 9$

or in specifying domain and range of a function

eg. $f : \mathbb{R} \to \mathbb{R}$

f^2 vs. $f^{(2)} = f''$

$\pi = 3.1415926\ldots$ vs. $\pi = 3.1415926$

Series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots$ $a_i \in \mathbb{R}$ (<u>numerical</u> series)

Every series has two sequences:

- the sequence of <u>terms</u> $a_1, a_2, a_3, a_4, \ldots$
- the sequence of <u>partial sums</u> $S_1, S_2, S_3, S_4, \ldots$ where $S_n = a_1 + a_2 + a_3 + \cdots + a_n$.

When we say that the series converges, we mean that the partial sums converge.

If $(S_n) \to L$ then we say the series converges to L and we write $\sum_{n=1}^{\infty} a_n = L$.

If a series does not converge, then we say it <u>diverges</u>.

The series $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots$

has n^{th} partial sum $s_n = a_0 + a_1 + a_2 + \cdots + a_{n-1}$ (the sum of the first n terms of the series). Here the n^{th} term is a_{n-1}. (This point is implicitly assumed in the video.)

Eg. the <u>harmonic series</u> $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ diverges.

The sequence of terms $\left(\frac{1}{n}\right) \to 0$ but the sequence of partial sums $(s_n) \to \infty$

where $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. (Note: $s_n \sim \ln n$)

Eg. $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$ diverges.

The sequence of partial sums is $1, 0, 1, 0, 1, 0, 1, 0, \cdots$ which diverges.

Eg. $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ converges.

the sequence of partial sums is $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \cdots$ which converges to 2 so

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

For series $\sum_{n=1}^{\infty} a_n$ with positive (or non-negative) terms, $a_n \geq 0$, only two things can happen with $s_n = a_1 + a_2 + a_3 + \cdots + a_n$

either $(s_n) \to \infty$

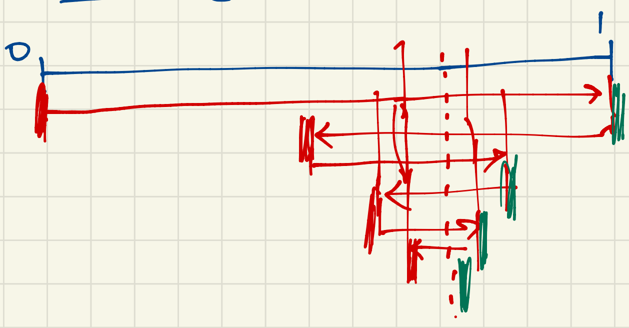
or $(s_n) \to L$.

This is a consequence of the Least Upper Bound Property.

We have $s_1 \leq s_2 \leq s_3 \leq s_4 \leq \cdots$ since $a_n \geq 0$ for all n.

By the Monotone Convergence Theorem, either (s_n) is unbounded and $(s_n) \to \infty$; or it converges, say $(s_n) \to L$ and $\sum a_n = L$. Here $L \geq 0$.

The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ converges.



Why does $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$? Here is a heuristic argument (<u>not</u> a complete proof)

≈ 0.693

Start with $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \quad (|x| < 1)$

$$\int_0^x \frac{1}{1-t}\,dt = \int_0^x (1 + t + t^2 + t^3 + t^4 + \cdots)\,dt = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \quad \text{for } (|x|<1)$$

$$-\ln(1-t)\Big|_0^x = -\ln(1-x)$$

Take the limit for both sides as $x \to 1^-$.

$$-\ln 2 = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \cdots$$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Is this justified?

In $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$ we can't let $x \to 1^-$, otherwise you get

$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$ which diverges.

For series with some positive and some negative terms, we cannot rearrange the series in general (permute the terms) without changing the answer.

Eg. The alternating harmonic series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots$ can be rearranged eg. as

$$1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}+\frac{1}{7}+\frac{1}{9}-\frac{1}{4}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}+\frac{1}{17}-\frac{1}{6}+\cdots$$

to make the series converge to anything you want (or to make it diverge).

Recall (from Calc II): If (a_n) is a decreasing sequence with $(a_n)\to 0$. Then $a_n \geq 0$ $(a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0)$ then $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots$ converges for the same reason as the alternating harmonic series converges. (The Alternating Series Test also known as the Leibniz Test for convergence).

A series $\sum_n a_n$ converges absolutely if $\sum_n |a_n|$ converges. In this case $\sum a_n$ converges and $|\sum a_n| \leq \sum |a_n|$.

If $\sum_n |a_n|$ diverges but $\sum_n a_n$ converges, then we say $\sum_n a_n$ converges conditionally.

Note: Every series can either

- converge absolutely, or
- converge conditionally, or
- diverge.

$\Big\}$ here $\sum a_n$ converges

Eg. $\sum_n \frac{1}{2^n}$ converges absolutely. So does $\sum \frac{(-1)^n}{2^n}$.

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges conditionally. (Note: $\sum_n \frac{1}{n}$ diverges.)

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Every conditionally convergent series can be rearranged to yield a convergent series with any sum you want, or to give a divergent series.

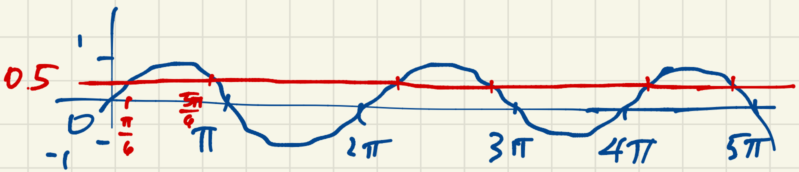
If a series converges absolutely, then every rearrangement will converge to the same value.

For series with terms ≥ 0, the series either converges absolutely or it diverges (conditional convergence cannot occur in this case).

$\sum_{n=1}^{\infty} \frac{\sin n}{n}$ is an example of a series with positive and negative terms but it is not an alternating series. Its terms are not monotone.

Note that $\sum \frac{\sin n}{n}$ cannot converge absolutely: $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n}\right|$ diverges.

In every interval $\left[2k\pi + \frac{\pi}{6},\ 2k\pi + \frac{5\pi}{6}\right]$ $(k = 0, 1, 2, 3, 4, 5, \ldots)$, $\sin x \geq \frac{1}{2}$. Each such interval has width $\frac{5\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{3} > 1$, there is at least one integer $n \geq 1$. This contributes terms $\geq \frac{1/2}{n}$ to the series. The sum $\sum_{n=1}^{\infty} \frac{|\sin n|}{n}$ which diverges. (Some details here omitted.)



$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \sum_{n=1}^{\infty} \frac{a^n - a^{-n}}{2i} \quad \text{where } a = e^i.$$

Recall: $e^{i\theta} = \cos\theta + i \sin\theta$

$e^{-i\theta} = \cos\theta - i \sin\theta$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{1}{2i} \sum_{n=1}^{\infty} \frac{a^n - a^{-n}}{n}$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

$$-\ln(1-a) = a + \frac{a^2}{2} + \frac{a^3}{3} + \frac{a^4}{4} + \frac{a^5}{5} + \cdots = \sum_{n=1}^{\infty} \frac{a^n}{n}$$

$$-\ln(1-a^{-1}) = a^{-1} + \frac{a^{-2}}{2} + \frac{a^{-3}}{3} + \frac{a^{-4}}{4} + \frac{a^{-5}}{5} + \cdots = \sum_{n=1}^{\infty} \frac{a^{-n}}{n}$$

$$\ln(1-a^{-1}) - \ln(1-a) = \sum_{n=1}^{\infty} \frac{a^n - a^{-n}}{n} = \frac{1}{2i} \sum_{n=1}^{\infty} \frac{\sin n}{n}$$

$$= \ln \frac{1-a^{-1}}{1-a} = \ln\left(\frac{1-a^{-1}}{1-a} \cdot \frac{a}{a}\right) = \ln\left(\frac{a-1}{a(1-a)}\right) = \ln\left(-\frac{1}{a}\right) = \ln(-a^{-1})$$

$$= \ln(-1) + \ln(a^{-1})$$

$$= \pi i - i$$

$$= (\pi - 1) i$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{(\pi - 1) i}{2i} = \frac{\pi - 1}{2}$$

$\ln a = i, \quad \ln(a^{-1}) = -i$

$a = e^{i}$

$a^{-1} = e^{-i}$

$e^{\pi i} = \cos\pi + i \sin\pi$

$e^{\pi i} = -1$

$$\boxed{e^{\pi i} + 1 = 0}$$

Euler's Formula

$\ln(-1) = \pi i$

$e^{\pi i} = -1$

$e^{(2k+1)\pi i} = -1$

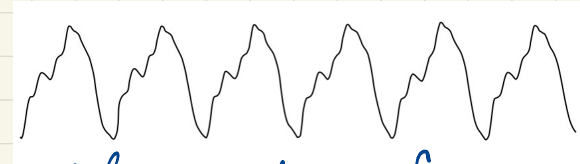
$\ln(-1) = (2k+1)\pi, \quad k \in \mathbb{Z}.$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \quad (1)$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \quad (2)$$

$$\sin(A-B) = \sin A \cos B - \sin B \cos A \quad (1')$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B \quad (2')$$



violin acoustic waveform

(1) + (1'): $\sin A \cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$

(2) + (2'): $\cos A \cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$

(2') − (2): $\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$

eg. $$f(x) = \underbrace{\underbrace{\sin(5x)}_{\text{odd}}\,\underbrace{\sin(2x)}_{\text{odd}}}_{\text{even: } f(-x) = f(x)} = -\frac{1}{2}\cos 7x - \frac{1}{2}\cos 3x$$

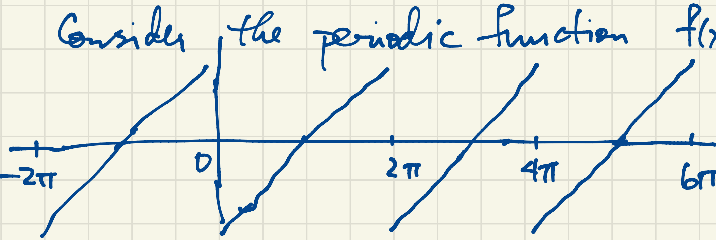
This is an example of a Fourier expansion.

$f(x+2\pi) = f(x)$ (periodic of period 2π)

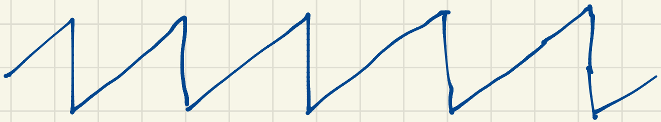
A periodic function f with period L satisfies $f(x+L) = f(x)$. In my examples $L = 2\pi$. Every periodic function $f(x)$ with period 2π has a Fourier expansion

$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]$$

Consider the periodic function $f(x) = x - \pi$ whenever $2k\pi < x < 2(k+1)\pi$, $k \in \mathbb{Z}$



Eg. For $0 < x < \pi$, $f(x) = x - \pi$.



Given f periodic with $f(x+2\pi) = f(x)$, we want to decompose $f(x)$ as

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right] = \sum_{n=1}^{\infty} \left(-\frac{2}{n}\right) \sin nx = -2 \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

Multiply both sides by $\sin(mx)$, then integrate from 0 to 2π.

$$\int_0^{2\pi} f(x) \sin mx\,dx = \int_0^{2\pi} \frac{a_0}{2} \sin mx\,dx + \sum_{n=1}^{\infty} \int_0^{2\pi} \left[a_n \cos(nx) \sin(mx) + b_n \sin(nx) \sin(mx) \right] dx$$

$$\frac{a_0}{2} \int_0^{2\pi} \sin(mx)\,dx = 0$$

$0 \qquad 0 \qquad \pi b_m$

$$\int_0^{2\pi} \cos(nx)\sin(mx)\,dx = \int_0^{2\pi} \frac{1}{2}\Big[\sin((m+n)x) + \sin((m-n)x)\Big]\,dx = 0$$

$$\int_0^{2\pi} \sin(nx)\sin(mx)\,dx = \frac{1}{2}\int_0^{2\pi} \Big[\cos((n-m)x) - \cos((n+m)x)\Big]\,dx = \begin{cases} 0 & \text{if } m \neq n \\ & \end{cases}$$

$m, n \geq 0$ If $m \neq n$ then we get 0.

If $m = n \neq 0$ then we get $\frac{1}{2}\int_0^{2\pi} \Big[1 - \cancel{\cos(2nx)}\Big]\,dx = \frac{1}{2} \cdot 2\pi = \pi$

If $m = n = 0$ then we get $\frac{1}{2}\int_0^{2\pi} \Big[1 - 1\Big]\,dx = 0$.

Solve: $b_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(mx)\,dx$. Similarly, $a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(mx)\,dx$.

For our function $f(x) =$  ... from the previous slide, let's work out its Fourier expansion. We should get only sine terms since the cosine terms are even.

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(mx)\,dx = \frac{1}{\pi}\int_0^{2\pi}(x-\pi)\cos(mx)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos mx\,dx - \int_0^{2\pi}\cos mx\,dx.$$

If $m \neq 0$, $\displaystyle\int_0^{2\pi} \underbrace{x}_{u}\,\underbrace{\cos mx\,dx}_{dv} = \frac{1}{m}\cancel{x\sin mx}\Big|_0^{2\pi} - \frac{1}{m}\cancel{\int_0^{2\pi}\sin mx\,dx} = 0$ so $a_m = 0.$

$u = x$

$du = dx$

$v = \frac{1}{m}\sin mx$

$dv = \cos mx\,dx$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}(x-\pi)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \pi x\right]_0^{2\pi} = \frac{1}{\pi}\left[\cancel{\frac{4\pi^2}{2} - 2\pi}\right] - \frac{1}{\pi}[0-0] = 0.$$

$$b_m = \frac{1}{\pi}\int_0^{2\pi}(x-\pi)\sin(mx)\,dx = \frac{1}{\pi}\int_0^{2\pi}\underbrace{x}_{u}\,\underbrace{\sin(mx)\,dx}_{dv} - \cancel{\int_0^{2\pi}\sin mx\,dx}$$

$(m \neq 0)$

$$\int u\,dv = uv - \int v\,du$$

$$\int \underbrace{x}_{u}\,\underbrace{\sin(mx)dx}_{v} = -\frac{1}{m}x\cos mx + \frac{1}{m}\int\cos(mx)\,dx$$

$u = x$ $\quad v = -\frac{1}{m}\cos(mx)$

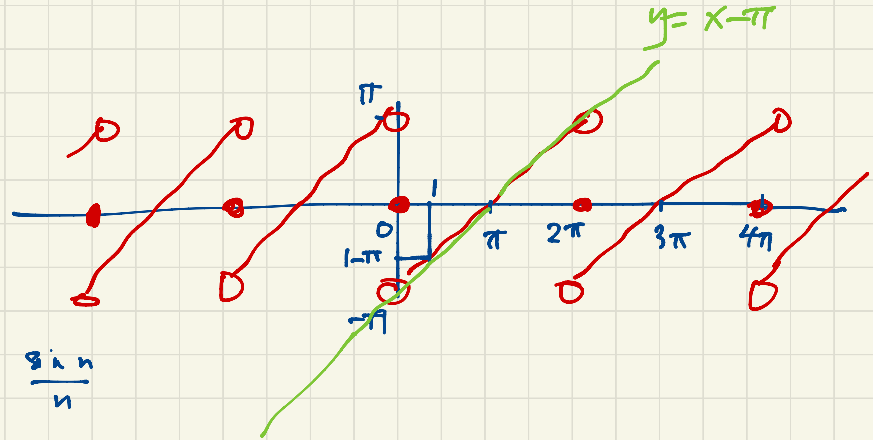
$du = dx$ $\quad dv = \sin(mx)\,dx$

$$b_m = \frac{1}{\pi}\left[-\frac{1}{m}x\cos mx + \frac{1}{m}\cancel{\int\cos(mx)\,dx}\right]_0^{2\pi} = \frac{1}{\pi}\left[-\frac{1}{m}\cdot 2\pi\cdot 1 + \frac{1}{m}\cdot 0\cdot 1\right] = -\frac{2}{m}$$

The Fourier series for $f(x)$ above is

$$f(x) \approx -2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

which converges for every $x \in \mathbb{R}$ to



Note: if $x = 1$, $f(1) = 1-\pi = -2\sum_{n=1}^{\infty} \frac{\sin n}{n}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\pi - 1}{2}$$

If $A = \{a_1, a_2, a_3, a_4, \cdots\}$ and $B = \{b_1, b_2, b_3, b_4, b_5, \cdots\}$ then $A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \cdots\}$. This shows that the union of two countable sets is countable.

If $A_1, A_2, A_3, \cdots$ are finite sets then $\bigcup_{n=1}^{\infty} A_n$ is countable.

Proof: Let $A_k = \{a_{k,1}, a_{k,2}, a_{k,3}, \cdots, a_{k,n_k}\}$.

Then $\bigcup_{n=1}^{\infty} A_n = \{a_{1,1}, a_{1,2}, \cdots, a_{1,n_1}, a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n_2}, a_{3,1}, a_{3,2}, \cdots, a_{3,n_3}, \cdots\}$.

So a countable union of finite sets is countable.

In fact, a countable union of countable sets is countable. Why?

If $A_1 = \{a_{11}, a_{12}, a_{13}, a_{14}, \cdots\}$

$A_2 = \{a_{21}, a_{22}, a_{23}, a_{24}, \cdots\}$

$A_3 = \{a_{31}, a_{32}, a_{33}, a_{34}, \cdots\}$

$A_4 = \{a_{41}, a_{42}, a_{43}, a_{44}, \cdots\}$

etc.

Then $\bigcup_{n=1}^{\infty} A_n = \{a_{11}, a_{12}, a_{21}, a_{31}, a_{22}, a_{13}, a_{14}, a_{23}, a_{32}, a_{41}, a_{51}, \cdots\}$

Cauchy's Criterion

A sequence $(a_n)_n$ is Cauchy if for all $\varepsilon > 0$ there exists N such that

$$|a_m - a_n| < \varepsilon \quad \text{whenever } m, n > N.$$

Theorem A sequence (a_n) converges iff it is Cauchy.

Eg. $a = 17.32511276484413\cdots$ defines a real number.

It is the limit of the sequence $\underset{a_1}{10}, \underset{a_2}{17}, \underset{a_3}{17.3}, \underset{a_4}{17.32}, \underset{a_5}{17.325}, \underset{a_6}{17.3257}, \cdots$

Eg. $a = 17.3251127648441 3\cdots$ defines a real number.

It is the limit of the sequence $\underset{a_1}{\underset{\|}{10}}, \underset{a_2}{\underset{\|}{17}}, \underset{a_3}{\underset{\|}{17.3}}, \underset{a_4}{\underset{\|}{17.32}}, \underset{a_5}{\underset{\|}{17.325}}, \underset{a_6}{\underset{\|}{17.3257}} \ldots$

This is a Cauchy sequence.

For $m, n \geq 1$, $|a_m - a_n| < 10$.

For $m, n \geq 2$, $|a_m - a_n| \leq 1$.

For $m, n \geq 3$, $|a_m - a_n| \leq 0.1$

For $m, n \geq 4$, $|a_m - a_n| \leq 0.01$

etc.

For all $m, n \geq k$, $|a_m - a_n| \leq 10^{-k+2}$

Given $\varepsilon > 0$, choose k such that $10^{-k+2} < \varepsilon$. So for all $m, n \geq k$, $|a_m - a_n| < \varepsilon$.

This proves that the sequence is Cauchy. So the sequence converges to some $a \in \mathbb{R}$.

This number is denoted $a = 17.32511276484413\ldots$

Eg. $a_n = \frac{n}{n+1}$ for $n = 0, 1, 2, 3, \cdots$

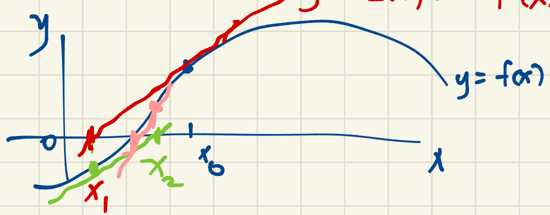
Prove that this sequence is Cauchy.

Proof: Let $\varepsilon > 0$. Then $|a_m - a_n| = \left|\frac{m}{m+1} - \frac{n}{n+1}\right| = \left|\left(1 - \frac{1}{m+1}\right) - \left(1 - \frac{1}{n+1}\right)\right|$

$= \left|\frac{1}{n+1} - \frac{1}{m+1}\right| \le \frac{1}{m+1} + \frac{1}{n+1} < \varepsilon$ whenever $m, n > \frac{2}{\varepsilon}$. $\square$

Remark (Rough work) If $m > \frac{2}{\varepsilon}$ then $m+1 > m > \frac{2}{\varepsilon}$ so $\frac{1}{m+1} < \frac{\varepsilon}{2}$.

Newton's Method is an iterative numerical approach to (hopefully) finding roots (solutions) of equations of the form $f(x) = 0$.



$0 = L(x_1) = f'(x_0)(x_1 - x_0) + f(x_0)$

$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

We consider the sequence of approximate roots $x_0, x_1, x_2, x_3, \cdots$ where x_0 is the initial guess;

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Eg. $f(x) = x^2 - 2$. Apply Newton's Method with $x_0 = 1$ to find an approximate value for $\sqrt{2}$. (Fact: The resulting sequence of iterates converges $(x_n) \to \sqrt{2}$.)

Here $f'(x) = 2x$. $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^2 - 2}{2x_n} = \dfrac{2x_n^2 - (x_n^2 - 2)}{2x_n} = \dfrac{x_n^2 + 2}{2x_n}$.

This gives $(x_n) \to \sqrt{2}$ as described in the video on "a Discrete Dynamical system". Here we iterate $g(x) = \dfrac{x^2 + 2}{2x}$.

In general, Newton's Method uses iteration of $g(x) = x - \dfrac{f(x)}{f'(x)}$.

a_n is a number (one term in a sequence or one element in a set

$(a_n)_{n \in \mathbb{N}}$ is a sequence i.e. $(a_1, a_2, a_3, a_4, \cdots) = (a_n : n \in \mathbb{N})$

$\{a_n\}_{n \in \mathbb{N}} = \{a_n : n \in \mathbb{N}\} = \{a_1, a_2, a_3, a_4, \cdots\}$

eg. $(1, 1, 1, 1, \cdots)$ is an infinite sequence whereas $\{1, 1, 1, 1, \cdots\} = \{1\}$ is a finite set.

See p. 89 or so.

accumulation point
cluster point

Let $A \subseteq \mathbb{R}$. A real number $b \in \mathbb{R}$ is a *limit point* of A if for every $\varepsilon > 0$, there exists $a \in A$ satisfying $0 < |a - b| < \varepsilon$. (This says that every small interval around b contains a point of A other than b.)

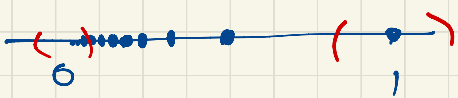
Note: Whether or not b is in A is irrelevant.

Eg. Every real number is a limit point of $\mathbb{Q}$. Given $\varepsilon > 0$, there exists a rational number in $(b, b+\varepsilon)$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. Such a rational $a \in \mathbb{Q} \cap (b, b+\varepsilon)$ satisfies $0 < |a-b| < \varepsilon$. This proves that every real number $b \in \mathbb{R}$ is a limit point of $\mathbb{Q}$.

Eg. $\mathbb{Z}$ has no limit points in $\mathbb{R}$. There is no element of $\mathbb{Z}$ that lies within $\frac{1}{2}$ of the number $\frac{1}{2}$. 0 is not a limit point of $\mathbb{Z}$

Note: 1 is not a limit point for the same reason as

Eg $\{\frac{1}{n} : n \geq 1\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$



0 is the only limit point of this set.