

# **Analysis I (Math 3205)**

## **Fall 2020**

**Book 3**

Let  $(a_n)_n$  be a sequence of real numbers. It is possible for such a sequence to have no limit point e.g.  $a_n = n$ . The sequence of positive integers has only isolated points. However, if  $(a_n)$  is bounded then it must have at least one limit point by the Bolzano-Weierstrass Theorem.

Eg. consider the sequence  $(\sin n)_{n \in \mathbb{N}} = (\sin 1, \sin 2, \sin 3, \sin 4, \dots)$ .

This sequence diverges. But the sequence is bounded (all terms lie in  $[-1, 1]$ )

So the sequence has a convergent subsequence. Thus there is at least one limit point. All limit points must lie in  $[-1, 1]$ .

$$\sin 0 = 0.000\dots$$

$$\sin 1 = 0.841\dots$$

$$\sin 2 = 0.909\dots$$

⋮

$$\sin 22 = -0.009$$

$$\sin 44 = 0.018$$

$$\sin 45 = 0.850$$

$$\sin 46 = 0.902$$

$$\pi \approx \frac{22}{7}$$

$$7\pi \approx 22$$

$$\sin 22 \approx \sin 7\pi = 0$$

$\sin n \neq 0$  for any positive integer  $n$  because  $\pi \notin \mathbb{Q}$ .

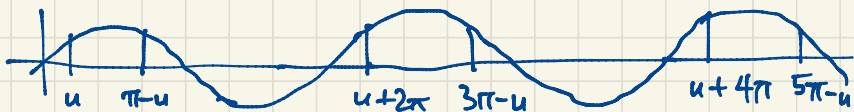
$$\sin x = 0 \Leftrightarrow x = k\pi \text{ for some } k \in \mathbb{Z}$$

Also since  $\pi \notin \mathbb{Q}$ , the sequence  $(\sin n)_n$  has no repeated terms

and the limit points of  $(\sin n)_n$  are all points of  $[-1, 1]$ .  $\left( \pi = \frac{n}{k} \in \mathbb{Q} \Leftrightarrow \sin n = 0 \Leftrightarrow n = k\pi \right.$  for some  $k \in \mathbb{Z}$

If  $\pi$  is irrational then the sequence  $(\sin u)_n$  has distinct terms (it never repeats).

Why? If  $\sin u = \sin v$  then either  $v - u = 2k\pi$  for some  $k \in \mathbb{Z}$   
 or  $v + u = (2k+1)\pi$  for some  $k \in \mathbb{Z}$ .



So if  $\sin m = \sin n$  where  $m \neq n$  are integers then either  $m - n = 2k\pi$  with  $0 \neq k \in \mathbb{Z}$  so  $\pi = \frac{m-n}{2k} \in \mathbb{Q}$ ; or  $m+n = (2k+1)\pi$  for some  $k \in \mathbb{Z}$  so  $\pi = \frac{m+n}{2k+1} \in \mathbb{Q}$

again contradicting  $\pi \notin \mathbb{Q}$ .

Let's prove  $\pi \notin \mathbb{Q}$ . Warm-up: prove  $e \notin \mathbb{Q}$ .

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n. \quad \text{Recall: } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$$

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= 1 \times 2 = 2 \\ 3! &= 1 \times 2 \times 3 = 6 \\ 4! &= 1 \times 2 \times 3 \times 4 = 24 \end{aligned}$$

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$$

Suppose  $e \in \mathbb{Q}$ ; say  $e = \frac{a}{b}$  in lowest terms ( $a, b \in \mathbb{N}$ ,  $\gcd(a, b) = 1$ ).

$$\frac{a}{b} = e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$$

Suppose  $e \in \mathbb{Q}$ , say  $e = \frac{a}{b}$  in lowest terms ( $a, b \in \mathbb{N}$ ,  $\gcd(a, b) = 1$ ).

Multiply both sides by  $b! = 1 \times 2 \times 3 \times \dots \times (b-1)b$ .

$$\begin{aligned}
 b! \cdot \frac{a}{b} &= \overbrace{(b-1)!}^{\text{integer}} a = b! \left( 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{(b-1)!} + \frac{1}{b!} + \frac{1}{(b+1)!} + \dots \right) \\
 &= b! + b! + \frac{b!}{2!} + \frac{b!}{3!} + \dots + \frac{b!}{(b-1)!} + \frac{b!}{b!} + \frac{b!}{(b+1)!} + \frac{b!}{(b+2)!} + \dots \\
 &\quad \left[ \begin{array}{cccc} \parallel & \parallel & \parallel & \parallel \\ & b & 1 & \frac{1}{b+1} & \frac{1}{(b+1)(b+2)} \end{array} \right]
 \end{aligned}$$

integers Not an integer by comparison test

The series  $\frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \dots$  converges by comparison with

$$\frac{1}{b+1} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \frac{1}{(b+1)^4} + \dots = \frac{1/(b+1)}{1 - \frac{1}{b+1}} = \frac{1}{(b+1)-1} = \frac{1}{b} < 1.$$

$$a + ar + ar^2 + ar^3 + \dots = \frac{a}{1-r} \quad (\text{for } |r| < 1).$$

This is a contradiction. So  $e \notin \mathbb{Q}$ .

$$(uv)' = u'v + uv'$$

$$(uv)'' = (u'v + uv')' = u''v + u'v' + u'v' + uv'' = u''v + 2u'v' + uv''$$

$$(uv)''' = (u''v + 2u'v' + uv'')' = (u'''v + u''v') + 2(u''v' + u'v'') + (u'v''' + uv''') \\ = u'''v + 3u''v' + 3u'v'' + uv'''$$

$$(u+v)' = u+v$$

$$(u+v)^2 = u^2 + 2uv + v^2$$

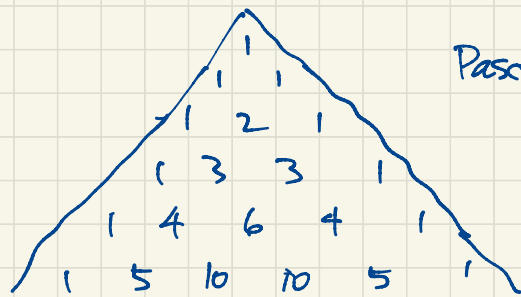
$$(u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$$

$$(u+v)^n = \sum_{k=0}^n \binom{n}{k} u^k v^{n-k}$$

(Binomial Theorem)

Leibniz' Formula

$$(uv)^{(n)} = \sum_{k=0}^n \binom{n}{k} u^{(k)} v^{(n-k)}$$



Pascal's Triangle

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \in \mathbb{Z}$$

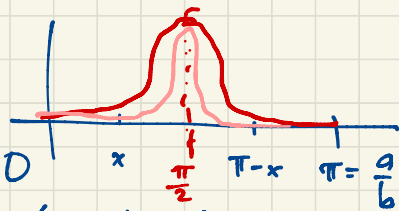
"binomial coefficients" are the entries in Pascal's Triangle

Theorem  $\pi \notin \mathbb{Q}$ .

Proof Suppose  $\pi = \frac{a}{b}$  in lowest terms (i.e.  $a, b \in \mathbb{N}$ ,  $\gcd(a, b) = 1$ ). We look for a contradiction. Consider the function  $f(x) = \frac{1}{n!} x^n (a - bx)^n$  where  $n \in \mathbb{N}$  will be chosen later. Note:  $f(x) = u(x)v(x)$  where  $u(x) = \frac{1}{n!} x^n$ ,  $v(x) = (a - bx)^n$ .

Lemma For every  $k \geq 0$ ,  $f^{(k)}(0) = (-1)^k f^{(k)}(\pi) \in \mathbb{Z}$ .

Proof  $f(\pi - x) = f\left(\frac{a}{b} - x\right) = \frac{1}{n!} \left(\frac{a}{b} - x\right)^n (a - b\left(\frac{a}{b} - x\right))^n = \frac{1}{n!} \left(\frac{a}{b} - x\right)^n (a - (a - bx))^n$   
 $= \frac{1}{n!} \left(\frac{a}{b} - x\right)^n (bx)^n = \frac{1}{n!} \left(\frac{a}{b} - x\right)^n b^n x^n = \frac{1}{n!} \left(\left(\frac{a}{b} - x\right)b\right)^n x^n = \frac{1}{n!} (a - bx)^n x^n = f(x)$ .



$$f(x) = \frac{1}{n!} (ax - bx^2)^n$$

$$f(\pi - x) = f(x)$$

$$-f'(\pi - x) = f'(x)$$

$$f''(\pi - x) = f''(x)$$

$$-f'''(\pi - x) = f'''(x)$$

$$f^{(k)}(x) = (-1)^k f^{(k)}(\pi - x)$$

$$f^{(k)}(0) = (-1)^k f^{(k)}(\pi)$$

We must show this  $\in \mathbb{Z}$ .

$$f(x) = u(x)v(x), \quad u(x) = \frac{1}{n!} x^n$$
$$u^{(k)}(0) = \begin{cases} 0 & \text{if } k \neq n; \\ 1 & \text{if } k = n. \end{cases}$$
$$\in \mathbb{Z}$$

$$u(x) = \frac{1}{n!} x^n$$

$$u'(x) = \frac{1}{n!} n x^{n-1} = \frac{1}{(n-1)!} x^{n-1}$$

$$u''(x) = \frac{1}{(n-2)!} x^{n-2}$$

$$\dots \quad u^{(n-2)}(x) = \frac{1}{2!} x^2 = \frac{1}{2} x^2$$

$$u^{(n-1)}(x) = x \quad u^{(n+1)}(x) = 0$$

$$u^{(n)}(x) = 1 \quad u^{(n+2)}(x) = 0 \text{ etc}$$

Recall:  $f(x) = u(x)v(x)$ ,  $u(x) = \frac{1}{a!}x^a$ ,  $v(x) = (a-bx)^n \in \mathbb{Z}[x]$

$$f^{(k)}(x) = \sum_{r=0}^k \binom{k}{r} u^{(r)}(x) v^{(k-r)}(x) \quad \text{[i.e. a polynomial in } x \text{ with integer coefficients]}$$

$$f^{(k)}(0) = \sum_{r=0}^k \underbrace{\binom{k}{r}}_{\text{integers}} \underbrace{u^{(r)}(0)}_{\text{integers}} \underbrace{v^{(k-r)}(0)}_{\text{integers}} \in \mathbb{Z}. \quad \text{This proves the lemma.}$$

Return to the Theorem.

Note:  $f(x)$  is a poly. in  $x$  of degree  $2n$ .

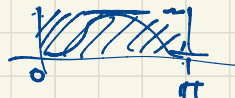
$$F'(x) = f''(x) - f^{(4)}(x) + f^{(6)}(x) - \dots + (-1)^{n-1} f^{(2n)}(x)$$

$$\text{Consider } F(x) = f(x) - f''(x) + f^{(4)}(x) - f^{(6)}(x) + \dots + (-1)^n f^{(2n)}(x).$$

$$\begin{aligned} \frac{d}{dx} [F'(x) \sin x - F(x) \cos x] &= F''(x) \sin x + F'(x) \cos x - (F'(x) \cos x - F(x) \sin x) \\ &= [F''(x) + F(x)] \sin x = F(x) \sin x \end{aligned}$$

$$\int_0^\pi f(x) \sin x \, dx = [F'(x) \sin x - F(x) \cos x]_0^\pi = F(\pi) - F(0) = F(b) - F\left(\frac{a}{b}\right) \in \mathbb{Z} \text{ by the lemma}$$

$$0 < \int_0^\pi f(x) \sin x \, dx < \pi \cdot f\left(\frac{\pi}{2}\right) = \frac{\pi}{n!} \left(\frac{\pi}{2}\right)^{2n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$



$f(x) = \frac{1}{n!}(ax-bx^2)^n$  is maximized at  $x = \frac{a}{2b} = \frac{a}{2b}$  on  $[0, \pi]$ . For  $n$  sufficiently large the integral is in  $(0, 1)$ , it can't be an integer.  $\square$

## Topology

Recall: If  $A \subseteq \mathbb{R}^n$ , a limit point  $a \in \mathbb{R}^n$  of  $A$  is a point such that for all  $\varepsilon > 0$ , there exists  $a \in A$  satisfying  $0 < |a - b| < \varepsilon$ .

The derived set of  $A$  is  $A' =$  the set of all limit points of  $A$ . Note: Limit points of  $A$  can belong to  $A$  but they don't have to.



eg.  $[0, 1)' = [0, 1]$   
 $\mathbb{Z}' = \emptyset$   
 $\mathbb{Q}' = \mathbb{R}$

$$\overline{[0, 1)} = [0, 1]$$
$$\overline{\mathbb{Z}} = \mathbb{Z}$$
$$\overline{\mathbb{Q}} = \mathbb{R}$$

The closure of  $A$  is  $\overline{A} = A \cup A'$ .

Note:  $\overline{\overline{A}} = \overline{A}$ .



An open set in  $\mathbb{R}^n$  is a union of open balls.

In  $\mathbb{R}$ , an open ball  $B_r(a) = \{x \in \mathbb{R} : |x-a| < r\}$  of radius  $r$  centered at  $a \in \mathbb{R}$  is the same thing as an open interval  $(a-r, a+r)$ .

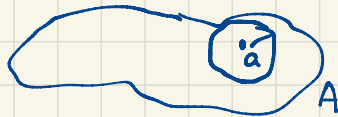
Every open interval  $(a, b)$  is an open set.  $(a, b) = B_{\frac{b-a}{2}}\left(\frac{a+b}{2}\right)$

Also  $(a, \infty) = \bigcup_{c > a} (c, c+1)$  is open.

$[0, 1]$  is not open.

$[0, 1)$  is not open. Proof: If  $[0, 1) = \bigcup_{i \in I} (a_i, b_i)$  for some collection of open intervals  $\{(a_i, b_i) : i \in I\}$  then  $0 \in (a_i, b_i)$  for some  $i \in I$ . Every such interval also contains some negative numbers, a contradiction.

Alternatively, a subset  $A \subseteq \mathbb{R}^n$  is open if every  $a \in A$  lies inside a ball  $B_\delta(a) \subseteq A$  for some  $\delta > 0$ .



A set  $A \subseteq \mathbb{R}$  is closed if it contains all its limit points (i.e.  $A' \subseteq A$  i.e.  $\bar{A} = A$ ) eg.  $[a, b]$  is closed.  $[a, b]' = [a, b]$ .

$$\overline{[a, b]} = [a, b] \cup [a, b]' = [a, b].$$

$[a, \infty)$  is closed.  $[a, \infty)' = [a, \infty)$ ,  $\overline{[a, \infty)} = [a, \infty) \cup [a, \infty)' = [a, \infty)$

$\bar{A}$  is the smallest closed set containing  $A$ .

$\mathbb{Z}$  is closed.

$\mathbb{Q}$  is not closed.  $\bar{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}' = \mathbb{Q} \cup \mathbb{R} = \mathbb{R}$ .

$\mathbb{R}$  is not open. ( $0 \in \mathbb{Q}$  is not covered by any  $B_\delta(0) = (-\delta, \delta)$  for  $\delta > 0$  inside  $\mathbb{Q}$ .)

Let  $A \subseteq \mathbb{R}^n$ . Then  $A$  is open iff its complement  $\mathbb{R}^n - A$  is closed.

eg.  $\mathbb{Z}$  is closed.  $\mathbb{R} - \mathbb{Z} = \bigcup_{n \in \mathbb{Z}} (n, n+1) = \dots \cup (-2, -1) \cup (-1, 0) \cup (0, 1) \cup (1, 2) \cup (2, 3) \cup \dots$  is open.

eg.  $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right\}$  is neither open nor closed.

$A' = \{0\}$ .  $\bar{A} = A \cup A' = \left\{ 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right\}$  is closed.



Its complement is open:  $\mathbb{R} - \bar{A} = (-\infty, 0) \cup (1, \infty) \cup \left( \bigcup_{n=1}^{\infty} \left( \frac{1}{n+1}, \frac{1}{n} \right) \right)$   
 $= (-\infty, 0) \cup (1, \infty) \cup \left( \frac{1}{2}, 1 \right) \cup \left( \frac{1}{3}, \frac{1}{2} \right) \cup \left( \frac{1}{4}, \frac{1}{3} \right) \cup \left( \frac{1}{5}, \frac{1}{4} \right) \cup \dots$

Can a set be both open and closed?

$\emptyset$  is both open and closed (i.e. clopen)

$\mathbb{R}$  is clopen.

$\emptyset$  and  $\mathbb{R}$  are the only clopen sets in  $\mathbb{R}$ . This is an important theorem which forms the basis for the Intermediate Value Theorem. The proof uses the completeness of  $\mathbb{R}$ .

Let  $X$  be a set. (eg.  $\mathbb{R}$  or  $\mathbb{R}^n$ ). A topology on  $X$  is a collection  $\mathcal{T}$  of subsets of  $X$  (script  $T$ ) called the open sets, satisfying:

- $\emptyset, X \in \mathcal{T}$
- Whenever  $A, B \in \mathcal{T}$ , we have  $A \cap B \in \mathcal{T}$ .
- Whenever  $\{A_i : i \in I\} \subseteq \mathcal{T}$ ,  $\bigcup_{i \in I} A_i \in \mathcal{T}$ .

$\emptyset, X$  are open.

Unions of open sets are open.

Intersections of finitely many open sets are open.

$$\text{eg. } \bigcap_{n \in \mathbb{N}} (0, \frac{n+1}{n}) = (0, 2) \cap (0, \frac{3}{2}) \cap (0, \frac{4}{3}) \cap (0, \frac{5}{4}) \cap (0, \frac{6}{5}) \cap \dots = (0, 1] \text{ is not open.}$$

$\emptyset, X$  are closed.

Intersections of closed sets are closed.

Unions of finitely many closed sets are closed.

$$\text{eg. } \bigcup_{0 < \delta < 1} [0, \delta] = [0, 1) \text{ is not closed.}$$

Eg. The Cantor Set is closed.

$$C = [0, 1] \cap \left( [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \right) \cap \left( [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1] \right) \cap \dots \text{ is closed since}$$

it is an intersection of closed sets.

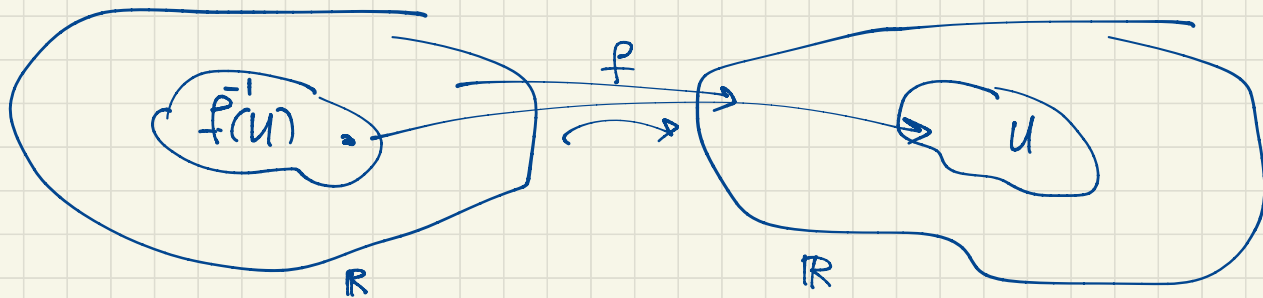
Recall:  $f: \mathbb{R} \rightarrow \mathbb{R}$  is **continuous** if for all  $a \in \mathbb{R}$  and  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|f(x) - f(a)| < \varepsilon$  whenever  $|x - a| < \delta$ .

The following statement is equivalent as a definition of continuity:

**For every open  $U \subseteq \mathbb{R}$ ,  $f^{-1}(U)$  is also open in  $\mathbb{R}$ .**

("The preimage of every open set is open").

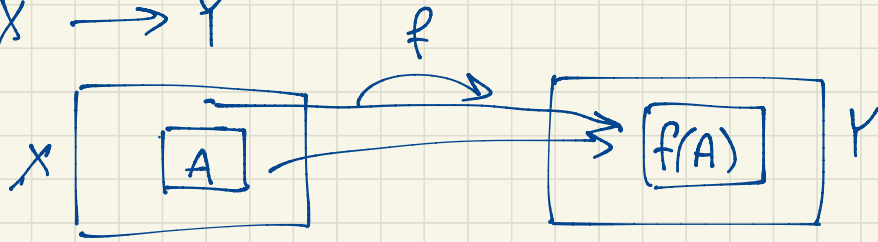
Note: We are not assuming  $f$  is one-to-one.  $f$  may not have an inverse function!



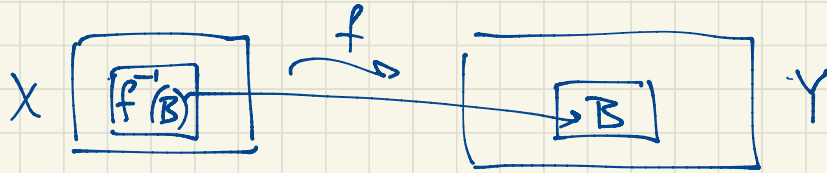
For every  $U \subseteq \mathbb{R}$ , define  $f^{-1}(U) = \{x \in \mathbb{R} : f(x) \in U\}$   
(the preimage of  $U$  under  $f$ )

Compare:  $f(U) = \{ f(u) : u \in U \}$  (the image of  $U$  under  $f$ )

If  $f: X \rightarrow Y$



$f^{-1}(f(A)) \supseteq A$  for all  $A \subseteq X$ .

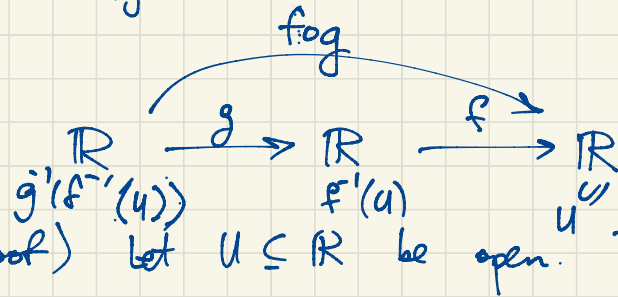


$f(f^{-1}(B)) \subseteq B$

eg.  $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$ .  $f^{-1}([4, 4]) = [2, 2]$   
 $f(f^{-1}([-4, 4])) = f([-2, 2]) = [0, 4] \subseteq [-4, 4]$

$f([0, 4]) = [0, 16]$   
 $f^{-1}(f([0, 4])) = f^{-1}([0, 16]) = [-4, 4] \supseteq [0, 4]$

Theorem: Suppose  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  are continuous. Then  $f \circ g: \mathbb{R} \rightarrow \mathbb{R}$  is continuous.

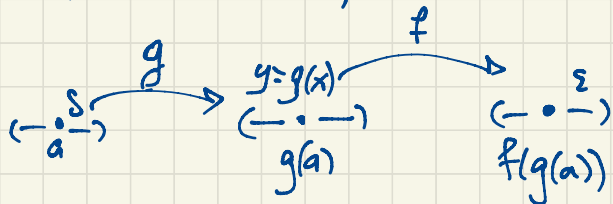


Proof (New proof) Let  $U \subseteq \mathbb{R}$  be open. Then

$$(f \circ g)^{-1}(U) = \underbrace{g^{-1}(f^{-1}(U))}_{\text{open}} \text{ is open because } \underbrace{f^{-1}(U)}_{\text{open}} \text{ is open.}$$

Compare:

(old proof) Let  $a \in \mathbb{R}, \varepsilon > 0$ . There exists  $\delta_1 > 0$  such that



$$|f(y) - f(g(a))| < \varepsilon \text{ whenever}$$

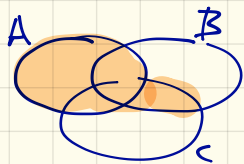
$$|y - g(a)| < \delta_1.$$

Also there exists  $\delta > 0$  such that  $|g(x) - g(a)| < \delta_1$  whenever  $|x - a| < \delta$ .

So whenever  $|x - a| < \delta$ , we have  $|f(g(x)) - f(g(a))| < \varepsilon$ .

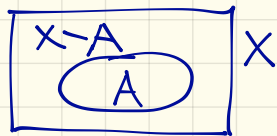


## Distributive Laws

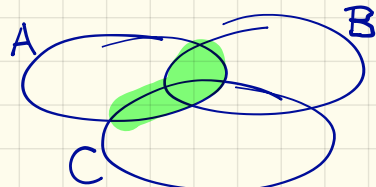


$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$



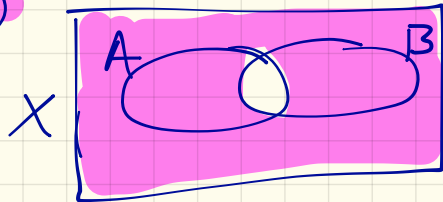
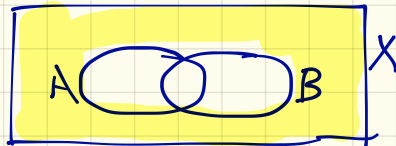
$$X - A = \{x \in X \mid x \notin A\}$$



De Morgan's Laws IF  $A, B \subseteq X$ ,

$$\rightarrow X - (A \cup B) = (X - A) \cap (X - B)$$

$$X - (A \cap B) = (X - A) \cup (X - B)$$



$\emptyset, X$  are open

$\{A_i \mid i \in I\}$  open sets  $\Rightarrow \bigcup_{i \in I} A_i$  open

$A_1, A_2, \dots, A_n$  open  $\Rightarrow A_1 \cap A_2 \cap \dots \cap A_n$  open

$\emptyset, X$  closed ( $X, \emptyset$  open)

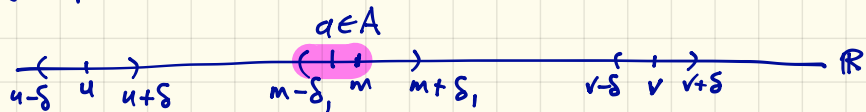
$\{K_i \mid i \in I\}$  closed sets  $\Rightarrow \bigcap_{i \in I} K_i = \bigcap_{i \in I} (X - U_i) = X - \bigcup_{i \in I} U_i$  is closed

$U_i = X - K_i$  open



Theorem The only clopen sets in  $\mathbb{R}$  are  $\emptyset$  and  $\mathbb{R}$

Proof Suppose  $U \neq \emptyset$ ,  $\mathbb{R}$  is clopen i.e.  $\mathbb{R} = U \sqcup V$  where  $U, V$  are disjoint nonempty open sets. Let  $u \in U, v \in V$  without loss of generality,  $u < v$



There exists  $\delta > 0$  such that  $(u-\delta, u+\delta) \subseteq U$  (since  $U$  is open) and  $(v-\delta, v+\delta) \subseteq V$

Let  $A$  be the set of all  $a \in [u, v]$  such that  $[u, a) \subseteq U$ . Clearly  $u+\delta \in A, v \notin A, [u, u+\delta) \subseteq A \subseteq [u, v-\delta]$ . Since  $A$  is a bounded nonempty subset of  $\mathbb{R}$  it has a least upper bound  $m = \sup A$  i.e.  $[u, m) \subseteq U$  but  $[u, a) \not\subseteq U$  for  $a > m$ .  $u+\delta \leq m \leq v-\delta$ . Note either  $m \in U$  or  $m \in V$ .

If  $m \in U$  then there exists  $\delta_1 > 0$  such that  $(m-\delta_1, m+\delta_1) \subseteq U$  (we make sure  $\delta_1 < \delta$ ) so that this interval stays inside  $[u, v]$ .

Since  $m = \sup A$ , there exists  $a \in A, m-\delta_1 \leq a \leq m$ . Then  $[u, a) \subseteq U, (m-\delta_1, m+\delta_1) \subseteq U$  so their union  $[u, m+\delta_1) \subseteq U$  so  $m+\delta_1 \in A, m+\delta_1 > m$  contradicting  $m = \sup A$ .

If  $m \in V$  we get a similar contradiction. □

Intermediate Value Theorem If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous taking some positive value and some negative value, then  $f(c) = 0$  for some  $c \in \mathbb{R}$

(Remark later we will consider functions  $f: [a, b] \rightarrow \mathbb{R}$  and even more general domains than this)

Proof Suppose  $f(\mathbb{R}) \subseteq (-\infty, 0) \cup (0, \infty)$  We must find a contradiction

Then  $\mathbb{R} = \underbrace{f^{-1}((-\infty, 0))}_{\{x \in \mathbb{R} \mid f(x) < 0\}} \cup \underbrace{f^{-1}((0, \infty))}_{\{x \in \mathbb{R} \mid f(x) > 0\}}$  is a disjoint union <sup>of two</sup> nonempty open sets, a contradiction.  $\square$

When we say the only clopen sets in  $\mathbb{R}$  are  $\emptyset$  and  $\mathbb{R}$ , this is saying  $\mathbb{R}$  is connected  
 $\mathbb{Q}$  is not connected

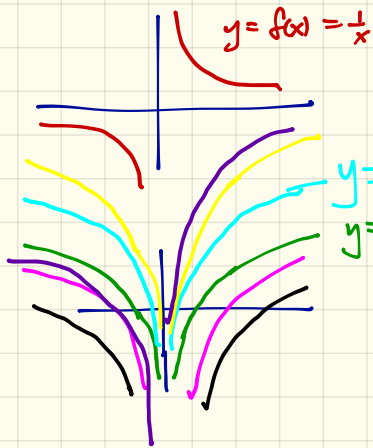
$$\mathbb{Q} = \{x \in \mathbb{Q} \mid x < \sqrt{2}\} \cup \{x \in \mathbb{Q} \mid x > \sqrt{2}\}$$

← nonempty open in  $\mathbb{Q}$

Recall An antiderivative for  $f$  is a function  $F$  such that  $F' = f$

what are the possible antiderivatives of  $f(x) = \frac{1}{x}$ ?

One antiderivative is  $\ln|x| = F(x)$



An antiderivative for  $f(x) = \frac{1}{x}$  is  
 $F(x) = \ln|x|$

Another  
 antiderivative is  
 $F(x) + 1$

A more general antiderivative is

$$F(x) + C = \ln|x| + C$$

where  $C \in \mathbb{R}$  is any constant

Are there others?

The general antiderivative (ie the most general antiderivative) for  $f$  is

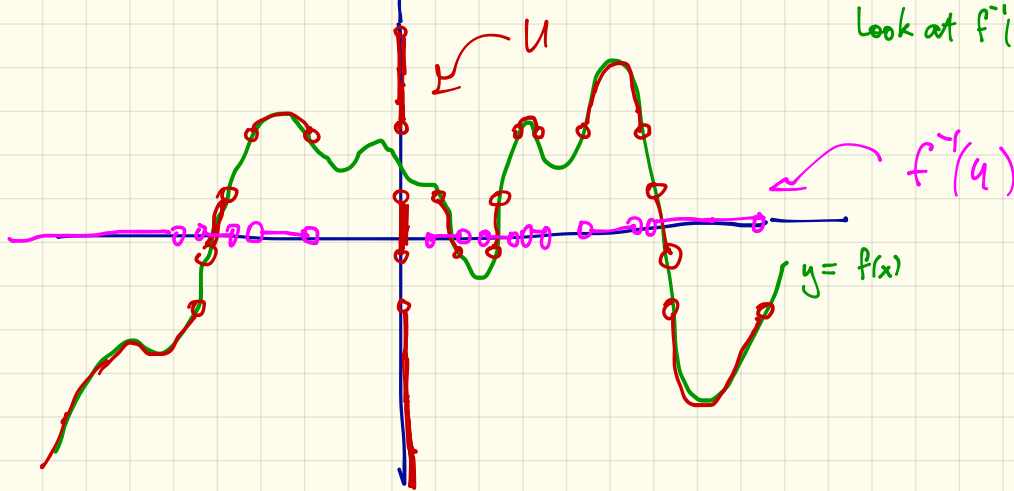
$$\begin{cases} \ln x + C_1 & \text{for } x > 0 \\ \ln|x| + C_2 & \text{for } x < 0 \end{cases}$$

where  $C_1, C_2 \in \mathbb{R}$  are arbitrary real constants

Why do we need more than one  
 arbitrary constant to express  
 the antiderivative of  $f$ ?

Because the domain of  $f$   
 is general won't be  
 connected

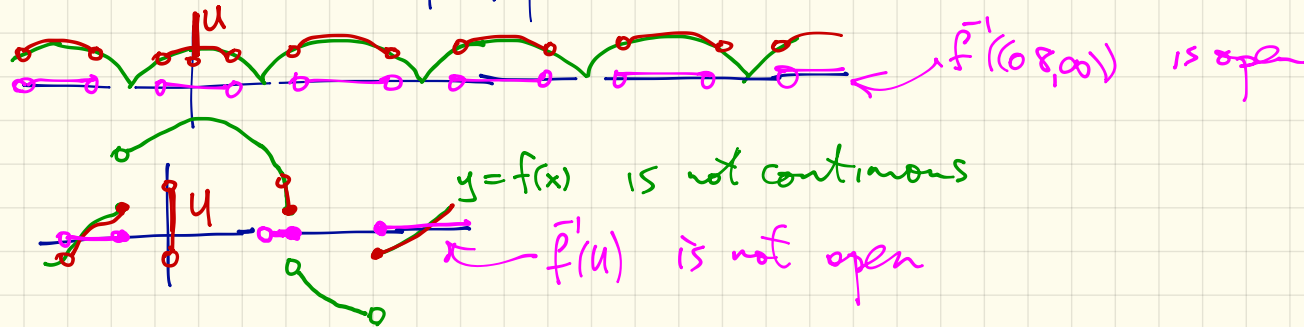
Look at  $f^{-1}(u)$  for a typical open set  $U$



$$= \{x \in \mathbb{R} \mid |\cos x| > 0.8\}$$

$$= \{x \mid f(x) \in (0.8, \infty)\}$$

Eg  $\{x \in \mathbb{R} \mid |\cos x| > 0.8\}$  is an open set in  $\mathbb{R}$  since it is  $f^{-1}((0.8, \infty))$   
 where  $f(x) = |\cos x|$



Big theorem from Calculus I

closed bounded interval

Every continuous function  $f: [a, b] \rightarrow \mathbb{R}$  has a maximum and a minimum

$$f(x) = e^x, x \in \mathbb{R}$$

has no maximum and no minimum

$[0, \infty)$  is a closed unbounded interval

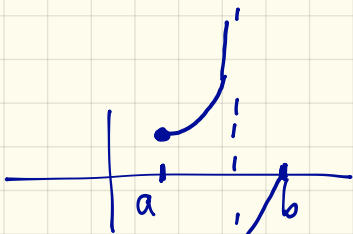
Eg



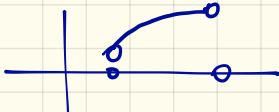
$f$  is continuous

$f$  is not bounded above

$f$  is bounded below, with 0 as a lower bound (0 is the greatest lower bound, i.e. the infimum of  $f$ ) 0 is not a value of  $f$  so it's certainly not a minimum value



Here is a discontinuous function defined on  $[a, b]$  with a minimum but no maximum



Here is a continuous function on an open interval with no maximum and no minimum

The relevance of  $[a, b]$  is that this is a compact set

Let's say what it means for a set  $A \subseteq \mathbb{R}$  (or  $\mathbb{R}^n$ ) to be compact

An open cover of  $A$  is a collection of open sets  $\{U_i, i \in I\}$  covering  $A$ , i.e.

$$A \subseteq \bigcup_{i \in I} U_i$$



It may often happen that a given open cover has a smaller subcover, i.e.  $\{U_i, i \in I'\}$ ,  $I' \subseteq I$  such that  $A \subseteq \bigcup_{i \in I'} U_i$ . Such a subcollection is called an open subcover.

We say  $A$  is compact if every open cover of  $A$  has a finite subcover.

$\mathbb{R}$  is not compact. It has an open cover consisting of all open intervals  $(a, a+1)$  of length 1. This has no finite subcover.

$\{2, 5, 9\} \subset \mathbb{R}$  is compact.

Heine-Borel Theorem  $[0, 1]$  is compact

Theorem Given  $A \subseteq \mathbb{R}$ , the following conditions are equivalent:

- (i)  $A$  is compact (ie. every open cover of  $A$  has a finite subcover)
- (ii)  $A$  is closed and bounded
- (iii)  $A$  is sequentially compact ie. every sequence in  $A$  has a subsequence converging in  $A$ . (This means converging to a point of  $A$ ).

The equivalence (i)  $\Leftrightarrow$  (iii) is by the Heine-Borel Theorem. The equivalence (i)  $\Leftrightarrow$  (ii) is another theorem.

Advice for doing mathematics:

- When you encounter a new topic/definition/theorem, put it to the test using examples.
- Make sure you learn the examples, not just the theorems.
- Don't start by paraphrasing.
- When learning a new topic, trust that the author/book/content is useful, beautiful, valid, coherent, etc.

Eg.  $\mathbb{Z}$  is not compact.  $\{(a, a+1) : a \in \mathbb{R}\}$  is an open cover of  $\mathbb{Z}$ .

Every finite subcollection  $\{(a_i, a_i+1) : i=1, 2, \dots, n\}$  is bounded and

$$\bigcup_{i=1}^n (a_i, a_i+1) \subseteq (r, s) \text{ where } r = \min\{a_1, \dots, a_n\}, \\ s = \max\{a_1, \dots, a_n\} + 1.$$

which is bounded so it doesn't cover  $\mathbb{Z}$ . Note that  $\mathbb{Z}$  is closed (it has no limit points) but not bounded.

Eg. The Cantor Set is compact. It is bounded (a subset of  $[0, 1]$ ) and it is closed.

Theorem Every closed subset of a compact set is compact.

Proof Let  $K$  be compact and let  $A \subseteq K$  be closed. So  $A' = \mathbb{R} - A$  is open.

Let  $\{U_i : i \in I\}$  be an open cover of  $A$ . (Thus  $U_i \subseteq \mathbb{R}$  is open for all  $i \in I$ ; and  $A \subseteq \bigcup_{i \in I} U_i$ .) Then  $\underbrace{\{U_i : i \in I\}}_{\text{covers } A} \cup \underbrace{\{A'\}}_{\text{covers } K-A}$  is an open cover of  $K$ .



Since  $K$  is compact, this open cover has a finite subcover  $\{U_1, U_2, \dots, U_n, A'\}$  so  $\{U_1, \dots, U_n\}$  covers  $A$ .  $\square$



Theorem Every <sup>nonempty</sup> compact set  $K \subseteq \mathbb{R}$  has a maximum and a minimum.

Think:  $(0, 1) \subset \mathbb{R}$  is not compact. It has no maximum or minimum.

Proof of the theorem: Let  $K \subseteq \mathbb{R}$  be a nonempty compact set.

So  $K$  is bounded. ( $K \subseteq \bigcup_{a \in \mathbb{R}} (-\infty, a) = \mathbb{R} \Rightarrow K \subseteq (-\infty, a_1) \cup \dots \cup (-\infty, a_n)$ )

for some  $a_1, \dots, a_n \in \mathbb{R}$  so  $K \subseteq (-\infty, a)$  where  $a = \max\{a_1, \dots, a_n\}$  so  $a$  is an upper bound for  $K$ .) So  $K$  has a least upper bound  $m = \sup K$ . I need to show  $m \in K$  (in which case it is the maximum element of  $K$ ). Continued on Tuesday...

If  $m \notin K$ ,  $K \subseteq \bigcup_{a < m} (-\infty, a)$ . (For every  $x \in K$ ,  $x \leq m$  so  $x < m$ .)

Since  $K$  is compact,

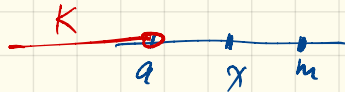
Pick  $a \in (x, m)$  so  $x \in (-\infty, a)$  where  $a < m$ .



so  $x \in \bigcup_{a < m} (-\infty, a)$ .

$K \subseteq (-\infty, a_1) \cup (-\infty, a_2) \cup \dots \cup (-\infty, a_n) = (-\infty, a)$  where  $a = \max\{a_1, a_2, \dots, a_n\} < m$ .

for some  $a_1, a_2, \dots, a_n < m$ . Pick  $x \in (a, m)$ . So  $x \notin K$ . In fact  $x$  is an upper bound for  $K \subseteq (-\infty, a)$



contradicting  $x < m$

where  $m$  is the least upper bound for  $K$ .

So  $m = \sup K \in K$

which must be the maximum element of  $K$ .

Theorem Let  $f: X \rightarrow Y$  be continuous. I will assume  $f$  is onto i.e. surjective (so  $Y$  is the image of  $f$  i.e.  $f(X) = Y$ .)

Remark:  $Y$  is usually called the range. But be careful: some books say "range" as a synonym for image.

- (i) If  $X$  is connected, then  $Y$  is connected.  
(ii) If  $X$  is compact, then so is  $Y$ .

Proof (i) Suppose  $Y$  is disconnected, then we must show  $X$  is disconnected. (This is the contrapositive.) If  $Y = U \sqcup V$  where  $U$  and  $V$  are open nonempty, then  $X = f^{-1}(U) \sqcup f^{-1}(V)$  where  $f^{-1}(U)$  and  $f^{-1}(V)$  are disjoint nonempty open sets.

(ii) Let  $\{U_i : i \in I\}$  be an open cover of  $Y$ . So  $U_i \subseteq Y$  is open for all  $i$ .

and  $Y \subseteq \bigcup_{i \in I} U_i$ . Then  $\{f^{-1}(U_i) : i \in I\}$  is an open cover of  $X$ .

$$X = f^{-1}(Y) \subseteq \bigcup_{i \in I} f^{-1}(U_i) \quad \left( \begin{array}{l} f(A \cup B) = f(A) \cup f(B) \\ A \subseteq B \Rightarrow f^{-1}(A) \subseteq f^{-1}(B) \end{array} \right)$$

Since  $X$  is compact,  $X \subseteq f^{-1}(U_{i_1}) \cup f^{-1}(U_{i_2}) \cup \dots \cup f^{-1}(U_{i_n})$  for some  $i_1, \dots, i_n \in I$ . So  $Y \subseteq U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_n}$ .

Intermediate Value Theorem If  $f: [a, b] \rightarrow \mathbb{R}$  is continuous with  $f(a) < 0 < f(b)$ , then  $f(c) = 0$  for some  $c \in (a, b)$ .

Proof  $f([a, b])$  is a connected subset of  $\mathbb{R}$ . (since it is the image of an interval  $[a, b]$  which is connected). See the videos on Topology. If  $0 \notin f([a, b])$  then  $f([a, b]) = U \sqcup V$  where  $U = f([a, b]) \cap (-\infty, 0)$ ,  $V = f([a, b]) \cap (0, \infty)$ .

$U, V$  are nonempty since  $f(a) \in U$ ,  $f(b) \in V$ . They are open subsets of the image.  $\therefore$  we have a continuous function  $f$  taking a connected domain  $[a, b]$  to a disconnected image  $f([a, b])$ , contradiction.  $\square$  **ASIDE**

Subspace topology: If  $A \subseteq \mathbb{R}$  then  $A$  inherits a topology from  $\mathbb{R}$ :

open sets in  $A$  are  $O \cap A$  where  $O \subseteq \mathbb{R}$  is open.

eg.  $\mathbb{Q} \subseteq \mathbb{R}$  is a subspace whose open sets look like  $O \cap \mathbb{Q}$  where  $O \subseteq \mathbb{R}$  is open.

$\mathbb{Q} = \mathbb{Q}_1 \sqcup \mathbb{Q}_2$ ,  $\mathbb{Q}_1 = \mathbb{Q} \cap (-\infty, \sqrt{2})$  is open in  $\mathbb{Q}$  (not in  $\mathbb{R}$  though)

$\mathbb{Q}_2 = \mathbb{Q} \cap (\sqrt{2}, \infty)$  is open in  $\mathbb{Q}$

So  $\mathbb{Q}$  is disconnected.

Theorem If  $f: [a, b] \rightarrow \mathbb{R}$  is continuous then  $f$  has a maximum and a minimum. (There exists  $c \in [a, b]$  such that  $f(x) \leq f(c)$  for all  $x \in [a, b]$ . Similarly for minimum.)  $c$  is a maximum point;  $f(c)$  is the maximum value.

Proof  $[a, b]$  is compact (by the Heine-Borel theorem) so  $f([a, b])$  is compact, hence closed and bounded. Also  $[a, b]$  is connected so  $f([a, b])$  is connected. So  $f([a, b])$  is an interval.

So  $f([a, b]) = [m_1, m_2]$  for some  $m_1, m_2 \in \mathbb{R}$ .

Then  $m_1$  is the minimum value of  $f$  and  $m_2$  is the maximum value of  $f$ . The values of  $f$  are all the numbers between  $m_1$  and  $m_2$ , inclusive.  $\square$

More generally if  $K \subseteq \mathbb{R}$  is compact then every continuous function  $f: K \rightarrow \mathbb{R}$  has a maximum and a minimum.

Let  $A \subseteq \mathbb{R}$ . Then  $A$  is connected iff  $A$  is an interval, i.e.

$(a, b)$ ,  $(a, b]$ ,  $[a, b)$ ,  $[a, b]$ ,  $[a, \infty)$ ,  $(-\infty, b)$ ,  $(-\infty, b]$ ,  $(-\infty, \infty) = \mathbb{R}$

---

Consider the sequence of functions  $f_n(x) = x^n$ ,  $0 \leq x \leq 1$ . These are continuous functions.  $(n = 1, 2, 3, \dots)$



$$\text{Let } f(x) = \lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0, & \text{for } 0 \leq x < 1; \\ 1, & \text{if } x = 1. \end{cases}$$

Note  $f_n \rightarrow f$  but  $f$  is discontinuous whereas  $f_n$  is continuous.

We have taken the limit  $f_n(x) \rightarrow f(x)$  pointwise (for each  $x \in [0, 1]$ )

The convergence is not uniform.

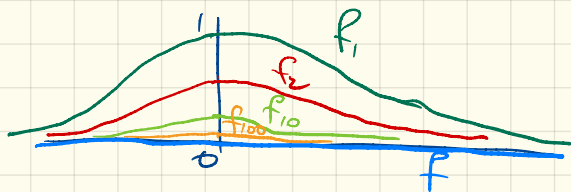
$\lim_{n \rightarrow \infty} f_n(x) = f(x)$  says: For all  $x \in [0, 1]$  and  $\varepsilon > 0$ , there exists  $N$  such that  $|f_n(x) - f(x)| < \varepsilon$  whenever  $n > N$ .

We take  $f_n(x)$  as a sequence of numbers for  $n = 1, 2, 3, \dots$  where  $x \in [0, 1]$  is fixed.

Note that  $N = N(\varepsilon, x)$ . The value of  $N$  will need to be larger if  $\varepsilon$  is taken as smaller; but also if  $x$  is taken as closer to 1.

We say  $f_n(x) \rightarrow f(x)$  converges uniformly for  $x \in A$  if  $N$  can be chosen independently of the choice of  $x \in A$  (i.e.  $N$  depends only on  $\varepsilon$ ).

Eg.  $f_n(x) = \frac{1}{n+x^2}$ , for  $x \in \mathbb{R}$ ,  $n \geq 1$ .



$f_n(x) \rightarrow f(x) = 0$  uniformly on  $\mathbb{R}$

For all  $\varepsilon > 0$  there exists  $N$  such that  $|f_n(x) - f(x)| < \varepsilon$  for all  $n > N$  and all  $x \in \mathbb{R}$ .

Here  $N = N(\varepsilon)$  is independent of the choice of  $x \in \mathbb{R}$  (it is chosen uniformly for the entire domain); it only depends on  $\varepsilon$ .

Here  $N = N(\varepsilon) = \frac{1}{\varepsilon}$ . If  $n > \frac{1}{\varepsilon}$  then  $|f_n(x) - \underbrace{f(x)}_0| = \frac{1}{n+x^2} \leq \frac{1}{n} < \varepsilon$ .

$$\text{Fig. } f_n(x) = \begin{cases} nx^2, & \text{if } 0 \leq x \leq \frac{1}{2n} \\ n^2(\frac{1}{2n} - x), & \text{if } \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 0, & \text{otherwise.} \end{cases}$$



$$\lim_{n \rightarrow \infty} f_n(x) = f(x) = 0 \quad \text{for all } x.$$

The convergence is not uniform



Here the limit of the continuous functions  $f_n$  is a continuous function  $f$ . But there is another problem:

$$\int_{-\infty}^{\infty} f_n(x) dx = \frac{1}{2} \cdot \frac{1}{n} \cdot 2n = 1, \quad \text{whereas} \quad \int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} 0 dx = 0.$$

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx \neq \int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx$$

$$\lim_{n \rightarrow \infty} 1 = 1 \quad \neq \quad \int_{-\infty}^{\infty} 0 dx = 0.$$

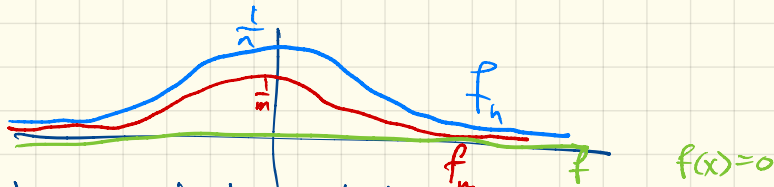
The failure of  $\lim \int f_n$  to equal  $\int \lim f_n$  is due to the fact that ~~our~~ convergence is not uniform.

Another way to view the distinction between pointwise and uniform convergence:  
 Define the distance between two functions  $f, g: A \rightarrow \mathbb{R}$  to be

$$d(f, g) = \|f - g\| \quad \text{where} \quad \|f\| = \sup_A |f| = \sup \{|f(a)| : a \in A\}.$$

Sometimes written as  $\|f\|_\infty$ . (Remark: It is usually preferable to ignore sets of measure zero in the domain.)

eg.  $f_n(x) = \frac{1}{n+x^2}$ ,  $n \in \mathbb{N}$



$$\|f_m - f_n\| = \sup_{x \in \mathbb{R}} |f_m(x) - f_n(x)| = \sup_{x \in \mathbb{R}} \left| \frac{1}{m+x^2} - \frac{1}{n+x^2} \right| = \frac{1}{m} - \frac{1}{n} \quad \text{if } m < n.$$

$$|g(x)| = f_m(x) - f_n(x) = \frac{1}{m+x^2} - \frac{1}{n+x^2} \quad \text{if } m < n$$

$$g'(x) = \frac{2x(m-n)(2x^2+m+n)}{(x^2+m)^2(x^2+n)^2}$$

$g'(x) < 0$  for  $x > 0$  i.e.  $g(x)$  is decreasing on  $(0, \infty)$

$g'(x) > 0$  for  $x < 0$  i.e.  $g(x)$  is increasing on  $(-\infty, 0)$ .

So  $g(x)$  has a unique maximum  $g(0) = \frac{1}{m} - \frac{1}{n}$ .

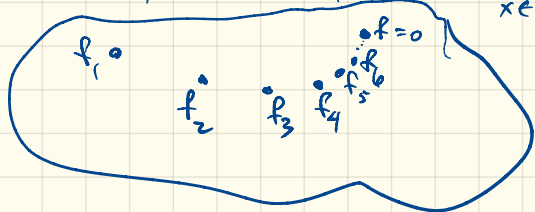
$$\|f_m - f_n\| = \left| \frac{1}{m} - \frac{1}{n} \right|$$



$f_n(x) = \frac{1}{n+x^2} \rightarrow f(x) = 0$  pointwise but also  $f_n \rightarrow f$  in the sup-norm i.e.

$d(f_n, f) \rightarrow 0$  as  $n \rightarrow \infty$ .

$$d(f_n, f) = \|f_n - f\| = \sup_{x \in \mathbb{R}} \left| \frac{1}{n+x^2} - 0 \right| = \frac{1}{n}$$



(STRONG CONVERGENCE)

Theorem If  $f_n \rightarrow f$  in norm then

$f_n(x) \rightarrow f(x)$  pointwise.

(But not conversely.)

To say  $f_n \rightarrow f$  in norm is much stronger than saying  $f_n(x) \rightarrow f(x)$  pointwise.

$f_n \rightarrow f$  in norm implies uniform convergence.

(WEAK CONVERGENCE)

Proof Let  $x \in A$  (the domain) and let  $\varepsilon > 0$ . Since  $f_n \rightarrow f$  in norm, there exists  $N = N(\varepsilon)$  (i.e. independent of  $x$ , i.e. uniformly for all  $x \in A$ ) such that  $\sup_{a \in A} |f_n(a) - f(a)| < \varepsilon$  for all  $n > N$ . Then

$$|f_n(x) - f(x)| \leq \sup_{a \in A} |f_n(a) - f(a)| < \varepsilon \text{ for all } n > N.$$

Thus  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ .

□

Why does the converse fail?

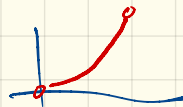
Consider  $f_n(x) = x^n$ ,  $0 \leq x \leq 1$ .

$f_n(x) \rightarrow f(x)$  pointwise on  $[0, 1]$  where  $f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1; \\ 1, & \text{if } x = 1. \end{cases}$

Does  $f_n \rightarrow f$  in norm? No:

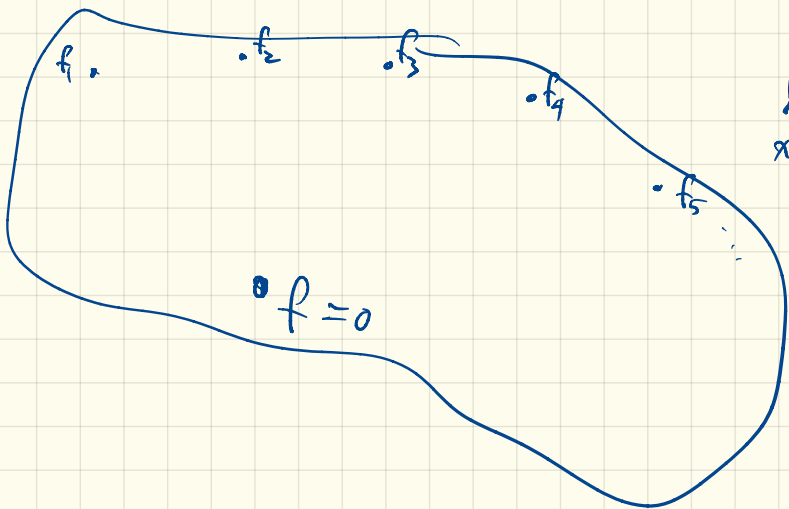
$$\|f_n - f\| = \sup_{x \in [0, 1]} |f_n(x) - f(x)| = 1 \quad \text{For } 0 < x < 1, |x^n - 0| = x^n$$

$$\sup_x x^n = 1.$$



$$\lim_{x \rightarrow 1^-} x^n = 1 \quad \text{for each } n \in \mathbb{N}.$$

The terms  $f_1, f_2, f_3, f_4, \dots$  do not approach  $f=0$ . They have distance 1 away from  $f$ .



Ex. The Taylor series for  $e^x$  is  $T(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots$

The partial sums are the Taylor polynomials  $T_n(x) = \sum_{k=0}^n \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots + \frac{x^n}{n!}$ .

For all  $x$ ,  $T(x)$  converges (absolutely) to  $e^x$ .

Actually,  $T(x) \rightarrow e^x$  pointwise on  $\mathbb{R}$ , not uniformly.

This means by definition that the sequence of partial sums  $T_n(x)$  converges pointwise to  $e^x$  on  $\mathbb{R}$ .

For all  $x$ ,  $\lim_{n \rightarrow \infty} T_n(x) = e^x$ . This convergence is not uniform on  $\mathbb{R}$ .

(How many terms do we need for  $T_n(x)$  to agree with  $e^x$  within  $\varepsilon$ ?)

This depends on how big  $x$  is. If  $x$  is close to zero, only a few terms are needed. For  $|x|$  large, many more terms in the series are needed.)

More concisely, the convergence  $T_n(x) \rightarrow e^x$  is uniform on  $[a, b]$  i.e. on compact subsets of  $\mathbb{R}$  but not on  $\mathbb{R}$ .