

Analysis I (Math 3205)

Fall 2020

Book 3

Let $(a_n)_n$ be a sequence of real numbers. It is possible for such a sequence to have no limit point e.g. $a_n = n$. The sequence of positive integers has only isolated points. However, if (a_n) is bounded then it must have at least one limit point by the Bolzano-Weierstrass Theorem.

Eg. consider the sequence $(\sin n)_{n \in \mathbb{N}} = (\sin 1, \sin 2, \sin 3, \sin 4, \dots)$.

This sequence diverges. But the sequence is bounded (all terms lie in $[-1, 1]$)

So the sequence has a convergent subsequence. Thus there is at least one limit point. All limit points must lie in $[-1, 1]$.

$$\sin 0 = 0.000\dots$$

$$\sin 1 = 0.841\dots$$

$$\sin 2 = 0.909\dots$$

⋮

$$\sin 22 = -0.009$$

$$\sin 44 = 0.018$$

$$\sin 45 = 0.850$$

$$\sin 46 = 0.902$$

$$\pi \approx \frac{22}{7}$$

$$7\pi \approx 22$$

$$\sin 22 \approx \sin 7\pi = 0$$

$\sin n \neq 0$ for any positive integer n because $\pi \notin \mathbb{Q}$.

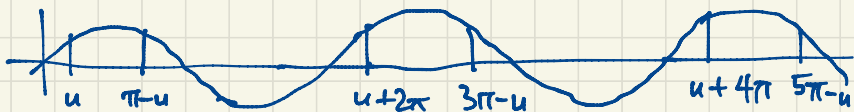
$$\sin x = 0 \Leftrightarrow x = k\pi \text{ for some } k \in \mathbb{Z}$$

Also since $\pi \notin \mathbb{Q}$, the sequence $(\sin n)_n$ has no repeated terms

and the limit points of $(\sin n)_n$ are all points of $[-1, 1]$. $\left(\pi = \frac{n}{k} \in \mathbb{Q} \Leftrightarrow \sin n = 0 \Leftrightarrow n = k\pi \right.$ for some $k \in \mathbb{Z}$

If π is irrational then the sequence $(\sin u)_n$ has distinct terms (it never repeats).

Why? If $\sin u = \sin v$ then either $v - u = 2k\pi$ for some $k \in \mathbb{Z}$
 or $v + u = (2k+1)\pi$ for some $k \in \mathbb{Z}$.



So if $\sin m = \sin n$ where $m \neq n$ are integers then either $m - n = 2k\pi$ with $0 \neq k \in \mathbb{Z}$ so $\pi = \frac{m-n}{2k} \in \mathbb{Q}$; or $m+n = (2k+1)\pi$ for some $k \in \mathbb{Z}$ so $\pi = \frac{m+n}{2k+1} \in \mathbb{Q}$

again contradicting $\pi \notin \mathbb{Q}$.

Let's prove $\pi \notin \mathbb{Q}$. Warm-up: prove $e \notin \mathbb{Q}$.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n. \quad \text{Recall: } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$$

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= 1 \times 2 = 2 \\ 3! &= 1 \times 2 \times 3 = 6 \\ 4! &= 1 \times 2 \times 3 \times 4 = 24 \end{aligned}$$

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$$

Suppose $e \in \mathbb{Q}$; say $e = \frac{a}{b}$ in lowest terms ($a, b \in \mathbb{N}$, $\gcd(a, b) = 1$).

$$\frac{a}{b} = e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$$

Suppose $e \in \mathbb{Q}$, say $e = \frac{a}{b}$ in lowest terms ($a, b \in \mathbb{N}$, $\gcd(a, b) = 1$).

Multiply both sides by $b! = 1 \times 2 \times 3 \times \dots \times (b-1)b$.

$$\begin{aligned}
 b! \cdot \frac{a}{b} &= \overbrace{(b-1)!}^{\text{integer}} a = b! \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{(b-1)!} + \frac{1}{b!} + \frac{1}{(b+1)!} + \dots \right) \\
 &= b! + b! + \frac{b!}{2!} + \frac{b!}{3!} + \dots + \frac{b!}{(b-1)!} + \frac{b!}{b!} + \frac{b!}{(b+1)!} + \frac{b!}{(b+2)!} + \dots \\
 &\quad \left[\begin{array}{cccc} \parallel & \parallel & \parallel & \parallel \\ & b & 1 & \frac{1}{b+1} \\ & & & \frac{1}{(b+1)(b+2)} \end{array} \right]
 \end{aligned}$$

Not an integer by comparison test

The series $\frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \dots$ converges by comparison with

$$\frac{1}{b+1} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \frac{1}{(b+1)^4} + \dots = \frac{1/(b+1)}{1 - \frac{1}{b+1}} = \frac{1}{(b+1)-1} = \frac{1}{b} < 1.$$

$$a + ar + ar^2 + ar^3 + \dots = \frac{a}{1-r} \quad (\text{for } |r| < 1).$$

This is a contradiction. So $e \notin \mathbb{Q}$.

$$(uv)' = u'v + uv'$$

$$(uv)'' = (u'v + uv')' = u''v + u'v' + u'v' + uv'' = u''v + 2u'v' + uv''$$

$$(uv)''' = (u''v + 2u'v' + uv'')' = (u'''v + u''v') + 2(u''v' + u'v'') + (u'v''' + uv''') \\ = u'''v + 3u''v' + 3u'v'' + uv'''$$

$$(u+v)' = u+v$$

$$(u+v)^2 = u^2 + 2uv + v^2$$

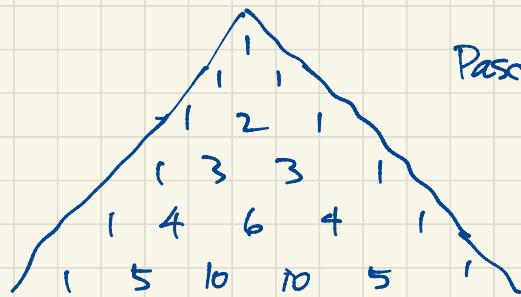
$$(u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$$

$$(u+v)^n = \sum_{k=0}^n \binom{n}{k} u^k v^{n-k}$$

(Binomial Theorem)

Leibniz' Formula

$$(uv)^{(n)} = \sum_{k=0}^n \binom{n}{k} u^{(k)} v^{(n-k)}$$



Pascal's Triangle

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \in \mathbb{Z}$$

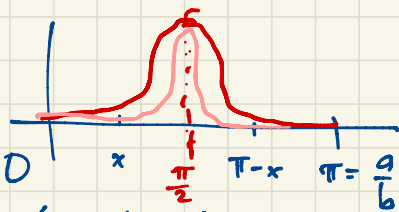
"binomial coefficients" are the entries in Pascal's Triangle

Theorem $\pi \notin \mathbb{Q}$.

Proof Suppose $\pi = \frac{a}{b}$ in lowest terms (i.e. $a, b \in \mathbb{N}$, $\gcd(a, b) = 1$). We look for a contradiction. Consider the function $f(x) = \frac{1}{n!} x^n (a - bx)^n$ where $n \in \mathbb{N}$ will be chosen later. Note: $f(x) = u(x)v(x)$ where $u(x) = \frac{1}{n!} x^n$, $v(x) = (a - bx)^n$.

Lemma For every $k \geq 0$, $f^{(k)}(0) = (-1)^k f^{(k)}(\pi) \in \mathbb{Z}$.

Proof $f(\pi - x) = f\left(\frac{a}{b} - x\right) = \frac{1}{n!} \left(\frac{a}{b} - x\right)^n (a - b\left(\frac{a}{b} - x\right))^n = \frac{1}{n!} \left(\frac{a}{b} - x\right)^n (a - (a - bx))^n$
 $= \frac{1}{n!} \left(\frac{a}{b} - x\right)^n (bx)^n = \frac{1}{n!} \left(\frac{a}{b} - x\right)^n b^n x^n = \frac{1}{n!} \left(\left(\frac{a}{b} - x\right)b\right)^n x^n = \frac{1}{n!} (a - bx)^n x^n = f(x)$.



$$f(x) = \frac{1}{n!} (ax - bx^2)^n$$

$$f(\pi - x) = f(x)$$

$$-f'(\pi - x) = f'(x)$$

$$f''(\pi - x) = f''(x)$$

$$-f^{(4)}(\pi - x) = f^{(4)}(x)$$

$$f^{(k)}(x) = (-1)^k f^{(k)}(\pi - x)$$

$$f^{(k)}(0) = (-1)^k f^{(k)}(\pi)$$

We must show this $\in \mathbb{Z}$.

$$f(x) = u(x)v(x), \quad u(x) = \frac{1}{n!} x^n$$
$$u^{(k)}(0) = \begin{cases} 0 & \text{if } k \neq n; \\ 1 & \text{if } k = n. \end{cases}$$
$$\in \mathbb{Z}$$

$$u(x) = \frac{1}{n!} x^n$$

$$u'(x) = \frac{1}{n!} n x^{n-1} = \frac{1}{(n-1)!} x^{n-1}$$

$$u''(x) = \frac{1}{(n-2)!} x^{n-2}$$

$$\dots \quad u^{(n-2)}(x) = \frac{1}{2!} x^2 = \frac{1}{2} x^2$$

$$u^{(n-1)}(x) = x \quad u^{(n+1)}(x) = 0$$

$$u^{(n)}(x) = 1 \quad u^{(n+2)}(x) = 0 \text{ etc}$$

Recall: $f(x) = u(x)v(x)$, $u(x) = \frac{1}{a^r}x^n$, $v(x) = (a-bx)^n \in \mathbb{Z}[x]$

$$f^{(k)}(x) = \sum_{r=0}^k \binom{k}{r} u^{(r)}(x) v^{(k-r)}(x) \quad \text{[i.e. a polynomial in } x \text{ with integer coefficients]}$$

$$f^{(k)}(0) = \sum_{r=0}^k \binom{k}{r} \underbrace{u^{(r)}(0)}_{\text{integers}} \underbrace{v^{(k-r)}(0)}_{\text{integers}} \in \mathbb{Z}. \quad \text{This proves the lemma.}$$

Return to the Theorem.

Note: $f(x)$ is a poly. in x of degree $2n$.

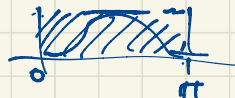
$$F'(x) = f''(x) - f^{(4)}(x) + f^{(6)}(x) - \dots + (-1)^{n-1} f^{(2n)}(x)$$

$$\text{Consider } F(x) = f(x) - f^{(4)}(x) + f^{(6)}(x) - f^{(8)}(x) + \dots + (-1)^n f^{(2n)}(x).$$

$$\begin{aligned} \frac{d}{dx} [F'(x) \sin x - F(x) \cos x] &= F''(x) \sin x + F'(x) \cos x - (F'(x) \cos x - F(x) \sin x) \\ &= [F''(x) + F(x)] \sin x = F(x) \sin x \end{aligned}$$

$$\int_0^\pi f(x) \sin x \, dx = [F'(x) \sin x - F(x) \cos x]_0^\pi = F(\pi) - F(0) = F(0) - F\left(\frac{\pi}{b}\right) \in \mathbb{Z} \text{ by the lemma}$$

$$0 < \int_0^\pi f(x) \sin x \, dx < \pi \cdot f\left(\frac{\pi}{2}\right) = \frac{\pi}{n!} \left(\frac{\pi}{2}\right)^{2n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$



$f(x) = \frac{1}{n!} (ax - bx^2)^n$ is maximized at $x = \frac{a}{2b} = \frac{a}{2b}$ on $[0, \pi]$. For n sufficiently large the integral is in $(0, 1)$, it can't be an integer. \square

Topology

Recall: If $A \subseteq \mathbb{R}^n$, a limit point a of A $b \in \mathbb{R}^n$ is a point such that for all $\varepsilon > 0$, there exists $a \in A$ satisfying $0 < |a - b| < \varepsilon$.

The derived set of A is $A' =$ the set of all limit points of A . Note: Limit points of A can belong to A but they don't have to.



eg. $[0, 1)' = [0, 1]$
 $\mathbb{Z}' = \emptyset$
 $\mathbb{Q}' = \mathbb{R}$

$\overline{[0, 1)} = [0, 1]$
 $\overline{\mathbb{Z}} = \mathbb{Z}$
 $\overline{\mathbb{Q}} = \mathbb{R}$

The closure of A is $\overline{A} = A \cup A'$.

Note: $\overline{\overline{A}} = \overline{A}$.

An open set in \mathbb{R}^n is a union of open balls.

In \mathbb{R} , an open ball $B_r(a) = \{x \in \mathbb{R} : |x-a| < r\}$ of radius r centered at $a \in \mathbb{R}$ is the same thing as an open interval $(a-r, a+r)$.

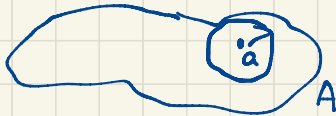
Every open interval (a,b) is an open set. $(a,b) = B_{\frac{b-a}{2}}\left(\frac{a+b}{2}\right)$

Also $(a, \infty) = \bigcup_{c > a} (c, c+1)$ is open.

$[0, 1]$ is not open.

$[0, 1)$ is not open. Proof: If $[0, 1) = \bigcup_{i \in I} (a_i, b_i)$ for some collection of open intervals $\{(a_i, b_i) : i \in I\}$ then $0 \in (a_i, b_i)$ for some $i \in I$. Every such interval also contains some negative numbers, a contradiction.

Alternatively, a subset $A \subseteq \mathbb{R}^n$ is open if every $a \in A$ lies inside a ball $B_\delta(a) \subseteq A$ for some $\delta > 0$.



A set $A \subseteq \mathbb{R}$ is closed if it contains all its limit points (i.e. $A' \subseteq A$ i.e. $\bar{A} = A$) eg. $[a, b]$ is closed. $[a, b]' = [a, b]$.

$$\overline{[a, b]} = [a, b] \cup [a, b]' = [a, b].$$

$[a, \infty)$ is closed. $[a, \infty)' = [a, \infty)$, $\overline{[a, \infty)} = [a, \infty) \cup [a, \infty)' = [a, \infty)$

\bar{A} is the smallest closed set containing A .

\mathbb{Z} is closed.

\mathbb{Q} is not closed. $\bar{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}' = \mathbb{Q} \cup \mathbb{R} = \mathbb{R}$.

\mathbb{R} is not open. ($0 \in \mathbb{Q}$ is not covered by any $B_\delta(0) = (-\delta, \delta)$ for $\delta > 0$ inside \mathbb{Q} .)

Let $A \subseteq \mathbb{R}^n$. Then A is open iff its complement $\mathbb{R}^n - A$ is closed.

eg. \mathbb{Z} is closed. $\mathbb{R} - \mathbb{Z} = \bigcup_{n \in \mathbb{Z}} (n, n+1) = \dots \cup (-2, -1) \cup (-1, 0) \cup (0, 1) \cup (1, 2) \cup (2, 3) \cup \dots$ is open.

eg. $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right\}$ is neither open nor closed.

$A' = \{0\}$. $\bar{A} = A \cup A' = \left\{ 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right\}$ is closed.



Its complement is open: $\mathbb{R} - \bar{A} = (-\infty, 0) \cup (1, \infty) \cup \left(\bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n} \right) \right)$
 $= (-\infty, 0) \cup (1, \infty) \cup \left(\frac{1}{2}, 1 \right) \cup \left(\frac{1}{3}, \frac{1}{2} \right) \cup \left(\frac{1}{4}, \frac{1}{3} \right) \cup \left(\frac{1}{5}, \frac{1}{4} \right) \cup \dots$

Can a set be both open and closed?

\emptyset is both open and closed (i.e. clopen)

\mathbb{R} is clopen.

\emptyset and \mathbb{R} are the only clopen sets in \mathbb{R} . This is an important theorem which forms the basis for the Intermediate Value Theorem. The proof uses the completeness of \mathbb{R} .

Let X be a set. (eg. \mathbb{R} or \mathbb{R}^n). A topology on X is a collection \mathcal{T} of subsets of X (script T) called the open sets, satisfying:

- $\emptyset, X \in \mathcal{T}$
- Whenever $A, B \in \mathcal{T}$, we have $A \cap B \in \mathcal{T}$.
- Whenever $\{A_i : i \in I\} \subseteq \mathcal{T}$, $\bigcup_{i \in I} A_i \in \mathcal{T}$.

\emptyset, X are open.

Unions of open sets are open.

Intersections of finitely many open sets are open.

$$\text{eg. } \bigcap_{n \in \mathbb{N}} (0, \frac{n+1}{n}) = (0, 2) \cap (0, \frac{3}{2}) \cap (0, \frac{4}{3}) \cap (0, \frac{5}{4}) \cap (0, \frac{6}{5}) \cap \dots = (0, 1] \text{ is not open.}$$

\emptyset, X are closed.

Intersections of closed sets are closed.

Unions of finitely many closed sets are closed.

$$\text{eg. } \bigcup_{0 < \delta < 1} [0, \delta] = [0, 1) \text{ is not closed.}$$

Eg. The Cantor Set is closed.

$$C = [0, 1] \cap \left([0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \right) \cap \left([0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1] \right) \cap \dots \text{ is closed since}$$

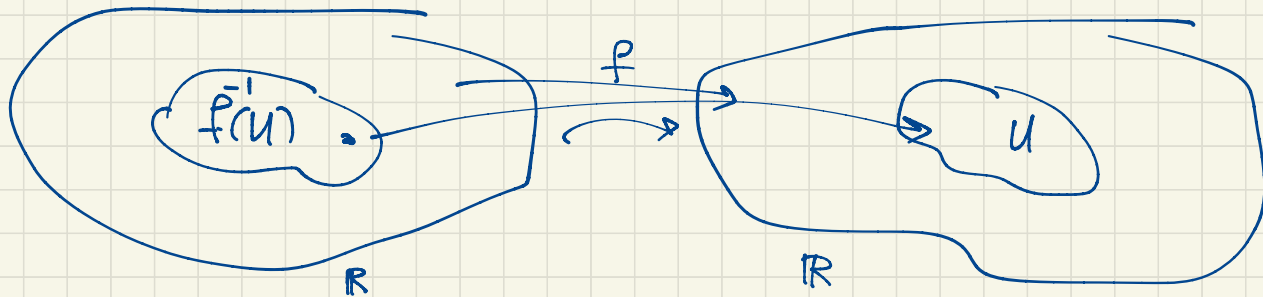
it is an intersection of closed sets.

Recall: $f: \mathbb{R} \rightarrow \mathbb{R}$ is **continuous** if for all $a \in \mathbb{R}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$.

The following statement is equivalent as a definition of continuity:

For every open $U \subseteq \mathbb{R}$, $f^{-1}(U)$ is also open in \mathbb{R} .
(“The preimage of every open set is open”).

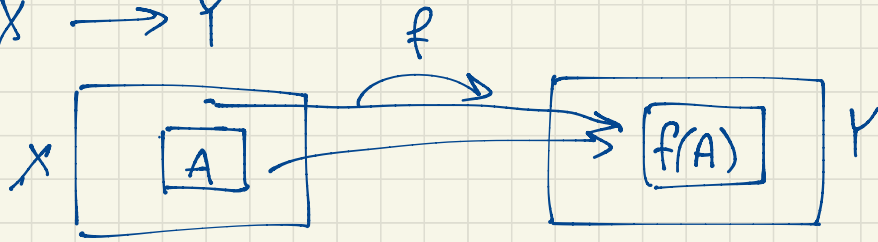
Note: We are not assuming f is one-to-one. f may not have an inverse function!



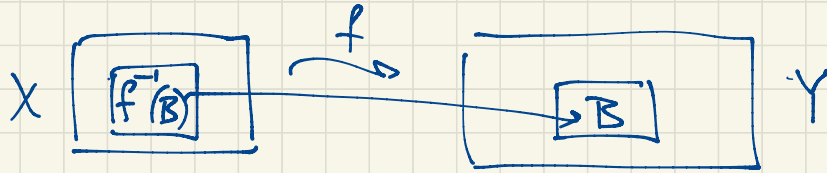
For every $U \subseteq \mathbb{R}$, define $f^{-1}(U) = \{x \in \mathbb{R} : f(x) \in U\}$
(the preimage of U under f)

Compare: $f(U) = \{ f(u) : u \in U \}$ (the image of U under f)

If $f: X \rightarrow Y$



$f^{-1}(f(A)) \supseteq A$ for all $A \subseteq X$.



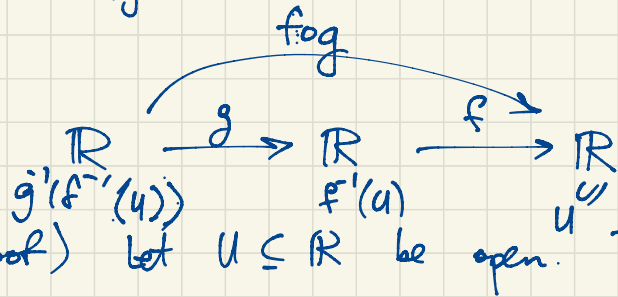
$f(f^{-1}(B)) \subseteq B$

eg. $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$. $f^{-1}([4, 4]) = [2, 2]$
 $f(f^{-1}([-4, 4])) = f([-2, 2]) = [0, 4] \subseteq [-4, 4]$

$$f([0, 4]) = [0, 16]$$

$$f^{-1}(f([0, 4])) = f^{-1}([0, 16]) = [-4, 4] \supseteq [0, 4]$$

Theorem: Suppose $f, g: \mathbb{R} \rightarrow \mathbb{R}$ are continuous. Then $f \circ g: \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

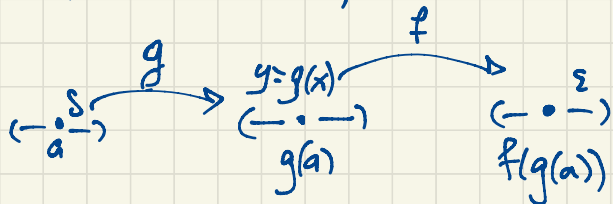


Proof (New proof) Let $U \subseteq \mathbb{R}$ be open. Then

$$(f \circ g)^{-1}(U) = \underbrace{g^{-1}(f^{-1}(U))}_{\text{open}} \text{ is open because } \underbrace{f^{-1}(U)}_{\text{open}} \text{ is open.}$$

Compare:

(old proof) Let $a \in \mathbb{R}, \varepsilon > 0$. There exists $\delta_1 > 0$ such that



$$|f(y) - f(g(a))| < \varepsilon \text{ whenever}$$

$$|y - g(a)| < \delta_1.$$

Also there exists $\delta > 0$ such that $|g(x) - g(a)| < \delta_1$ whenever $|x - a| < \delta$.

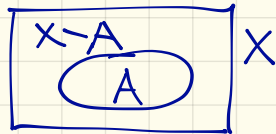
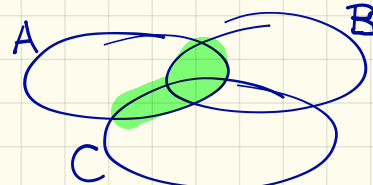
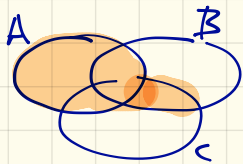
So whenever $|x - a| < \delta$, we have $|f(g(x)) - f(g(a))| < \varepsilon$.



Distributive Laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

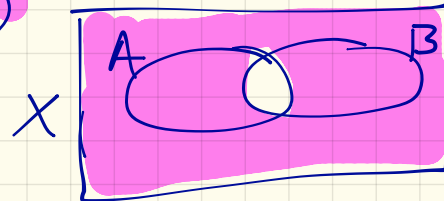
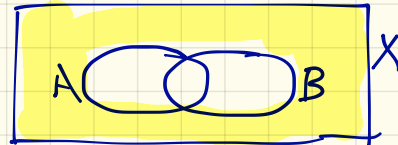


$$X - A = \{x \in X : x \notin A\}$$

De Morgan's Laws IF $A, B \subseteq X$,

$$\rightarrow X - (A \cup B) = (X - A) \cap (X - B)$$

$$X - (A \cap B) = (X - A) \cup (X - B)$$



\emptyset, X are open

$\{A_i : i \in I\}$ open sets $\Rightarrow \bigcup_{i \in I} A_i$ open

A_1, \dots, A_n open $\Rightarrow A_1 \cap A_2 \cap \dots \cap A_n$ open

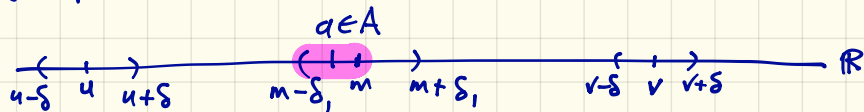
\emptyset, X closed (X, \emptyset open)

$\{K_i : i \in I\}$ closed sets $\Rightarrow \bigcap_{i \in I} K_i = \bigcap_{i \in I} (X - U_i) = X - \bigcup_{i \in I} U_i$ is closed

$U_i = X - K_i$ open

Theorem The only clopen sets in \mathbb{R} are \emptyset and \mathbb{R} .

Proof Suppose $U \neq \emptyset$, \mathbb{R} is clopen i.e. $\mathbb{R} = U \cup V$ where U, V are disjoint nonempty open sets. Let $u \in U, v \in V$. Without loss of generality, $u < v$.



There exists $\delta > 0$ such that $(u-\delta, u+\delta) \subseteq U$. (since U is open)
and $(v-\delta, v+\delta) \subseteq V$.

Let A be the set of all $a \in [u, v]$ such that $[u, a) \subseteq U$. Clearly $u+\delta \in A, v \notin A, [u, u+\delta) \subseteq A \subseteq [u, v-\delta]$. Since A is a bounded nonempty subset of \mathbb{R} it has a least upper bound $m = \sup A$. i.e. $[u, m) \subseteq U$ but $[u, a) \not\subseteq U$ for $a > m$.
 $u+\delta \leq m \leq v-\delta$. Note: either $m \in U$ or $m \in V$.

If $m \in U$ then there exists $\delta_1 > 0$ such that $(m-\delta_1, m+\delta_1) \subseteq U$ (we make sure $\delta_1 < \delta$)
so that this interval stays inside $[u, v]$.

Since $m = \sup A$, there exists $a \in A, m-\delta_1 \leq a \leq m$. Then $[u, a) \subseteq U, (m-\delta_1, m+\delta_1) \subseteq U$
so their union $[u, m+\delta_1) \subseteq U$ so $m+\delta_1 \in A, m+\delta_1 > m$ contradicting $m = \sup A$.

If $m \in V$ we get a similar contradiction. □

Intermediate Value Theorem If $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous taking some positive value and some negative value, then $f(c) = 0$ for some $c \in \mathbb{R}$.

(Remark: Later we will consider functions $f: [a, b] \rightarrow \mathbb{R}$ and even more general domains than this.)

Proof Suppose $f(\mathbb{R}) \subseteq (-\infty, 0) \cup (0, \infty)$. We must find a contradiction.

Then $\mathbb{R} = \underbrace{f^{-1}(-\infty, 0)}_{\{x \in \mathbb{R} : f(x) < 0\}} \sqcup \underbrace{f^{-1}(0, \infty)}_{\{x \in \mathbb{R} : f(x) > 0\}}$ is a disjoint union ^{of two} nonempty open sets, a contradiction. \square

When we say the only clopen sets in \mathbb{R} are \emptyset and \mathbb{R} , this is saying \mathbb{R} is connected.

\mathbb{Q} is not connected:

$$\mathbb{Q} = \{x \in \mathbb{Q} : x < \sqrt{2}\} \sqcup \{x \in \mathbb{Q} : x > \sqrt{2}\}$$

← nonempty open in \mathbb{Q} .