Analysis I (Math 3205)
Fall 2020

Book 3

Let $(a_n)_n$ be a sequence of real numbers. It is possible for such a sequence to have no limit point e.g. $a_n = n$. The sequence of positive integers has only isolated points. However, if (a_n) is bounded then it must have at least one limit point by the Bolzano-Weierstrass Theorem.

Eg. consider the sequence $(\sin n)_{n \in \mathbb{N}} = (\sin 1, \sin 2, \sin 3, \sin 4, \dots)$.

This sequence diverges. But the sequence is bounded (all terms lie in $[-1, 1]$)

So the sequence has a convergent subsequence. Thus there is at least one limit point. All limit points must lie in $[-1, 1]$.

$\sin 0 = 0.000\ldots$

$\sin 1 = 0.841\ldots$

$\sin 2 = 0.909\ldots$

$\vdots$

$\sin 22 = -0.009$

$\sin 44 = 0.018$

$\sin 45 = 0.850$

$\sin 46 = 0.902$

$\pi \approx \frac{22}{7}$

$7\pi \approx 22$

$\sin 22 \approx \sin 7\pi = 0$

$\sin n \neq 0$ for any positive integer n because $\pi \notin \mathbb{Q}$.

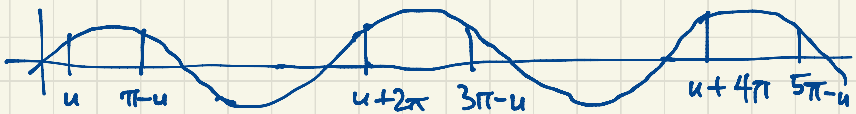
$\sin x = 0 \iff x = k\pi$ for some $k \in \mathbb{Z}$

Also since $\pi \notin \mathbb{Q}$, the sequence $(\sin n)_n$ has no repeated terms and the limit points of $(\sin n)_n$ are all points of $[-1, 1]$.

$\left(\pi = \frac{n}{k} \in \mathbb{Q} \right. \Leftarrow \sin n = 0 \iff n = k\pi$ for some $k \in \mathbb{Z}$

If π is irrational then the sequence $(\sin n)_n$ has distinct terms (it never repeats).

Why? If $\sin u = \sin v$ then either $v-u = 2k\pi$ for some $k\in\mathbb{Z}$
or $v+u = (2k+1)\pi$ for some $k\in\mathbb{Z}$.



So if $\sin m = \sin n$ where $m\neq n$ are integers then either $m-n = 2k\pi$ with $0\neq k\in\mathbb{Z}$ so $\pi = \frac{m-n}{2k}\in\mathbb{Q}$; or $m+n = (2k+1)\pi$ for some $k\in\mathbb{Z}$ so $\pi = \frac{m+n}{2k+1}\in\mathbb{Q}$ again contradicting $\pi\notin\mathbb{Q}$.

$0! = 1$
$1! = 1$
$2! = 1\times 2 = 2$

$3! = 1\times 2\times 3 = 6$
$4! = 1\times 2\times 3\times 4 = 24$

Let's prove $\pi\notin\mathbb{Q}$. Warm-up: prove $e\notin\mathbb{Q}$.

$$e = \sum_{n=0}^{\infty}\frac{1}{n!} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Recall: $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$

$$e = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots$$

Suppose $e\in\mathbb{Q}$, say $e = \frac{a}{b}$ in lowest terms ($a,b\in\mathbb{N}$, $\gcd(a,b)=1$).

$$\frac{a}{b} = e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

Suppose $e \in \mathbb{Q}$; say $e = \frac{a}{b}$ in lowest terms ($a, b \in \mathbb{N}$, $\gcd(a,b) = 1$).
Multiply both sides by $b! = 1 \times 2 \times 3 \times \cdots \times (b-1)b$.

$$b!\,\frac{a}{b} = \overbrace{(b-1)!\,a}^{\text{integer}} = b!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(b-1)!} + \frac{1}{b!} + \frac{1}{(b+1)!} + \cdots\right)$$

$$= \underbrace{b! + b! + \frac{b!}{2!} + \frac{b!}{3!} + \cdots + \underbrace{\frac{b!}{(b-1)!}}_{= b} + \underbrace{\frac{b!}{b!}}_{= 1}}_{\text{integers}} + \underbrace{\underbrace{\frac{b!}{(b+1)!}}_{= \frac{1}{b+1}} + \underbrace{\frac{b!}{(b+2)!}}_{= \frac{1}{(b+1)(b+2)}} + \cdots}_{\text{Not an integer by comparison test}}$$

The series $\frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \cdots$ converges by comparison with

$$\frac{1}{b+1} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \frac{1}{(b+1)^4} + \cdots = \frac{1/(b+1)}{1 - \frac{1}{b+1}} = \frac{1}{(b+1)-1} = \frac{1}{b} < 1.$$

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r} \quad (\text{for } |r| < 1).$$

This is a contradiction. So $e \notin \mathbb{Q}$.

$$(uv)' = u'v + uv'$$

$$(uv)'' = (u'v+uv')' = u''v + u'v' + u'v' + uv'' = u''v + 2u'v' + uv''$$

$$(uv)''' = (u''v+2u'v'+uv'')' = (u'''v + u''v') + 2(u''v'+u'v'') + (u'v''+uv''')$$

$$= u'''v + 3u''v' + 3u'v'' + uv'''$$

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$(u+v)^1 = u+v$$

$$(u+v)^2 = u^2 + 2uv + v^2$$

$$(u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$$

$$(u+v)^n = \sum_{k=0}^{n} \binom{n}{k} u^k v^{n-k}$$

(Binomial Theorem)

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \in \mathbb{Z}$$

"binomial coefficients" are the entries in Pascal's Triangle

Leibniz' Formula

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}$$

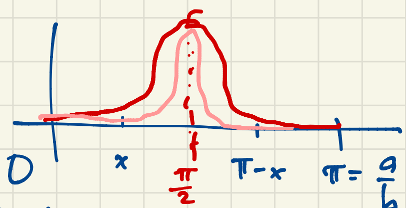
Theorem $\pi \notin \mathbb{Q}$.

Proof Suppose $\pi = \frac{a}{b}$ in lowest terms (i.e. $a, b \in \mathbb{N}$, $\gcd(a,b) = 1$). We look for a contradiction. Consider the function $f(x) = \frac{1}{n!} x^n (a-bx)^n$ where $n \in \mathbb{N}$ will be chosen later. Note: $f(x) = u(x) v(x)$ where $u(x) = \frac{1}{n!} x^n$, $v(x) = (a-bx)^n$.

Lemma For every $k \geq 0$, $f^{(k)}(0) = (-1)^k f^{(k)}(\pi) \in \mathbb{Z}$.

Proof $f(\pi - x) = f(\frac{a}{b} - x) = \frac{1}{n!}(\frac{a}{b} - x)^n (a - b(\frac{a}{b} - x))^n = \frac{1}{n!}(\frac{a}{b} - x)^n (a - (a - bx))^n$

$= \frac{1}{n!}(\frac{a}{b} - x)^n (bx)^n = \frac{1}{n!}(\frac{a}{b} - x)^n b^n x^n = \frac{1}{n!}((\frac{a}{b} - x)b)^n x^n = \frac{1}{n!}(a-bx)^n x^n = f(x)$.



$f(x) = \frac{1}{n!}(ax - bx^2)^n$

$$f(\pi - x) = f(x)$$
$$-f'(\pi - x) = f'(x)$$
$$f''(\pi - x) = f''(x)$$
$$-f'''(\pi - x) = f'''(x)$$

$$f^{(k)}(x) = (-1)^k f^{(k)}(\pi - x)$$
$$f^{(k)}(0) = (-1)^k f^{(k)}(\pi)$$

We must show this $\in \mathbb{Z}$.

$f(x) = u(x) v(x)$, $u(x) = \frac{1}{n!} x^n$

$u'(x) = \frac{1}{n!} \cdot n x^{n-1} = \frac{1}{(n-1)!} x^{n-1}$

$u''(x) = \frac{1}{(n-2)!} x^{n-2}$ ... $u^{(n-2)}(x) = \frac{1}{2!} x^2 = \frac{1}{2} x^2$

$u^{(n-1)}(x) = x$ $\qquad u^{(n+1)}(x) = 0$

$u^{(n)}(x) = 1$ $\qquad u^{(n+2)}(x) = 0$ etc.

$$u^{(k)}(0) = \begin{cases} 0 & \text{if } k \neq n; \\ 1 & \text{if } k = n. \end{cases}$$
$\in \mathbb{Z}$

Recall: $f(x) = u(x)v(x)$, $u(x) = \frac{1}{n!}x^n$, $\underbrace{v(x) = (a-bx)^n}$ $\in \mathbb{Z}[x]$

i.e. a polynomial in x with integer coefficients

$$f^{(k)}(x) = \sum_{r=0}^{k} \binom{k}{r} u^{(r)}(x)\, v^{(k-r)}(x)$$

$$f^{(k)}(0) = \sum_{r=0}^{k} \underbrace{\binom{k}{r}}_{\text{integers}} \underbrace{u^{(r)}(0)}_{\text{integers}}\, \underbrace{v^{(k-r)}(0)}_{\text{integers}} \in \mathbb{Z}.$$

This proves the Lemma.

Note: $f(x)$ is a poly. in x of degree $2n$.

Return to the Theorem.

Consider $F(x) = f(x) - f''(x) + f^{(4)}(x) - f^{(6)}(x) + \cdots + (-1)^n f^{(2n)}(x)$

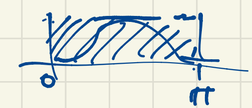
$F''(x) = f''(x) - f^{(4)}(x) + f^{(6)}(x) - \cdots + (-1)^{n-1} f^{(2n)}(x)$

$$\frac{d}{dx}\left[F'(x)\sin x - F(x)\cos x\right] = F''(x)\sin x + \cancel{F'(x)\cos x} - \left(\cancel{F'(x)\cos x} - F(x)\sin x\right)$$

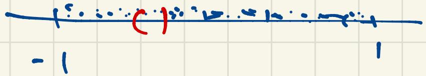
$$= \left[F''(x) + F(x)\right]\sin x = f(x)\sin x$$

$$\int_0^{\pi} f(x)\sin x\,dx = \left[\cancel{F'(x)\sin x} - F(x)\cos x\right]_0^{\pi} = F(\pi) - F(0) = F(0) - F(\tfrac{a}{b}) \in \mathbb{Z}$$ by the Lemma

$$0 < \int_0^{\pi} f(x)\sin x\,dx < \pi \cdot f(\tfrac{\pi}{2}) = \frac{\pi}{n!}\left(\frac{\pi}{2}\right)^{2n} \to 0 \text{ as } n\to\infty.$$



$f(x) = \frac{1}{n!}(ax - bx^2)^n$ is maximized at $x = \frac{\pi}{2} = \frac{a}{2b}$ on $[0,\pi]$. For n sufficiently large the integral is in $(0,1)$, it can't be an integer. $\square$



Topology

Recall: If $A \subseteq \mathbb{R}^n$, a limit point of A $b \in \mathbb{R}^n$ is a point such that for all $\varepsilon > 0$, there exists $a \in A$ satisfying $0 < |a - b| < \varepsilon$.

The derived set of A is A' = the set of all limit points of A. Note: Limit points of A can belong to A but they don't have to.

eg. $[0,1)' = [0,1]$ $\qquad$ $\overline{[0,1)} = [0,1]$

$\mathbb{Z}' = \emptyset$ $\qquad$ $\overline{\mathbb{Z}} = \mathbb{Z}$

$\mathbb{Q}' = \mathbb{R}$ $\qquad$ $\overline{\mathbb{Q}} = \mathbb{R}$

The closure of A is $\overline{A} = A \cup A'$.

Note: $\overline{\overline{A}} = \overline{A}$.

An open set in $\mathbb{R}^n$ is a union of open balls.

In $\mathbb{R}$, an open ball $B_r(a) = \{x \in \mathbb{R} : |x-a| < r\}$ of radius r centered at $a \in \mathbb{R}$ is the same thing as an open interval $(a-r, a+r)$.

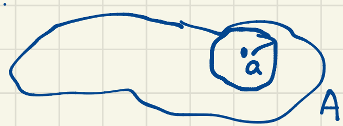
Every open interval (a,b) is an open set. $(a,b) = B_{\frac{b-a}{2}}\left(\frac{a+b}{2}\right)$

Also $(a,\infty) = \bigcup_{c \geq a} (c, c+1)$ is open.

$[0,1]$ is not open.

$[0,1)$ is not open. Proof: If $[0,1) = \bigcup_{i \in I} (a_i, b_i)$ for some collection of open intervals $\{(a_i, b_i) : i \in I\}$ then $0 \in (a_i, b_i)$ for some $i \in I$. Every such interval also contains some negative numbers, a contradiction.

Alternatively, a subset $A \subseteq \mathbb{R}^n$ is open if every $a \in A$ lies inside a ball $B_\delta(a) \subseteq A$ for some $\delta > 0$.



A set $A \subseteq \mathbb{R}$ is <u>closed</u> if it contains all its limit points (i.e. $A' \subseteq A$ i.e. $\overline{A} = A$) eg. $[a,b]$ is closed. $[a,b]' = [a,b]$.

$\overline{[a,b]} = [a,b] \cup [a,b]' = [a,b]$.

$[a,\infty)$ is closed. $[a,\infty)' = [a,\infty)$, $\overline{[a,\infty)} = [a,\infty) \cup [a,\infty)' = [a,\infty)$

$\overline{A}$ is the smallest closed set containing A.

$\mathbb{Z}$ is closed.

$\mathbb{Q}$ is not closed. $\overline{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}' = \mathbb{Q} \cup \mathbb{R} = \mathbb{R}$.

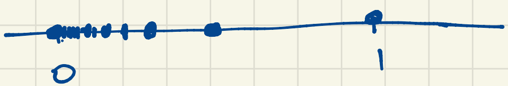
$\mathbb{Q}$ is not open. ($0 \in \mathbb{Q}$ is not covered by any $B_\delta(0) = (-\delta, \delta)$ for $\delta > 0$ inside $\mathbb{Q}$.)

Let $A \subseteq \mathbb{R}^n$. Then A is open iff its complement $\mathbb{R}^n - A$ is closed.

eg. $\mathbb{Z}$ is closed. $\mathbb{R} - \mathbb{Z} = \bigcup_{n \in \mathbb{Z}} (n, n+1) = \cdots \cup (-2,-1) \cup (-1,0) \cup (0,1) \cup (1,2) \cup (2,3) \cup \cdots$ is open.

eg. $A = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$ is neither open nor closed.

$A' = \{0\}$. $\quad \bar{A} = A \cup A' = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$ is closed.



$\bar{A}$

Its complement is open: $\mathbb{R} - \bar{A} = (-\infty, 0) \cup (1, \infty) \cup \left(\bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right) \right)$

$$= (-\infty, 0) \cup (1, \infty) \cup (\tfrac{1}{2}, 1) \cup (\tfrac{1}{3}, \tfrac{1}{2}) \cup (\tfrac{1}{4}, \tfrac{1}{3}) \cup (\tfrac{1}{5}, \tfrac{1}{4}) \cup \dots$$

Can a set be both open and closed?

$\emptyset$ is both open and closed (i.e. clopen)

$\mathbb{R}$ is clopen.

$\emptyset$ and $\mathbb{R}$ are the only clopen sets in $\mathbb{R}$. This is an important theorem which forms the basis for the Intermediate Value Theorem. The proof uses the completeness of $\mathbb{R}$.

Let X be a set. (eg. $\mathbb{R}$ or $\mathbb{R}^n$). A topology on X is a collection $\mathcal{T}$ of subsets of X (script T) called the open sets, satisfying:

- $\emptyset, X \in \mathcal{T}$
- whenever $A, B \in \mathcal{T}$, we have $A \cap B \in \mathcal{T}$.
- whenever $\{A_i : i \in I\} \subseteq \mathcal{T}$, $\bigcup_{i \in I} A_i \in \mathcal{T}$.

$\emptyset, X$ are open.
Unions of open sets are open.
Intersections of finitely many open sets are open.

eg. $\bigcap_{n \in \mathbb{N}} (0, \frac{n+1}{n}) = (0,2) \cap (0, \frac{3}{2}) \cap (0, \frac{4}{3}) \cap (0, \frac{5}{4}) \cap (0, \frac{6}{5}) \cap \cdots = (0, 1]$ is not open.

$\emptyset, X$ are closed.
Intersections of closed sets are closed.
Unions of finitely many closed sets are closed.

eg. $\bigcup_{0<\delta<1} [0, \delta] = [0, 1)$ is not closed.

Eg. The Cantor Set is closed.

$C = [0,1] \cap \left([0,\frac{1}{3}] \cup [\frac{2}{3}, 1]\right) \cap \left([0,\frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]\right) \cap \cdots$ is closed since it is an intersection of closed sets.

Recall: $f:\mathbb{R}\to\mathbb{R}$ is continuous if for all $a\in\mathbb{R}$ and $\varepsilon>0$, there exists $\delta>0$ such that

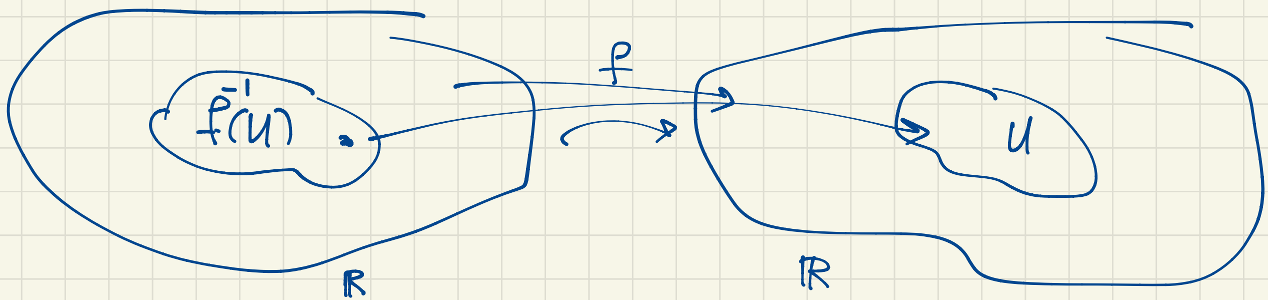
$|f(x)-f(a)|<\varepsilon$ whenever $|x-a|<\delta$.

The following statement is equivalent as a definition of continuity:

For every open $U\subseteq\mathbb{R}$, $f^{-1}(U)$ is also open in $\mathbb{R}$.

("The preimage of every open set is open").

Note: We are not assuming f is one-to-one. f may not have an inverse function!

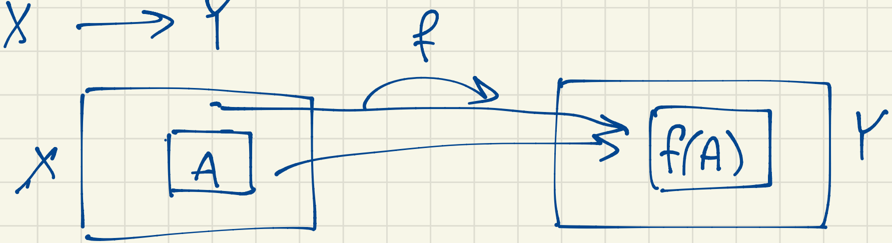


For every $U\subseteq\mathbb{R}$, define $f^{-1}(U)=\{x\in\mathbb{R} : f(x)\in U\}$

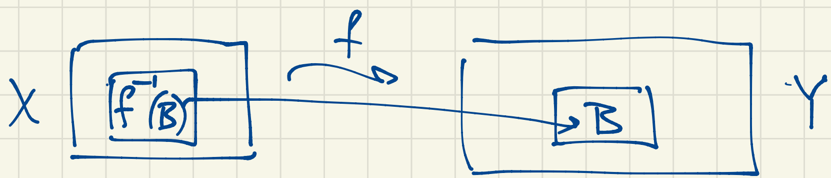
(the preimage of U under f).

Compare: $f(U) = \{ f(u) : u \in U \}$ (the image of U under f).

If $f : X \longrightarrow Y$



$f^{-1}(f(A)) \supseteq A$ for all $A \subseteq X$.



$f(f^{-1}(B)) \subseteq B$

eg. $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$.

$f^{-1}([-4,4]) = [-2,2]$

$f(f^{-1}([-4,4])) = f([-2,2]) = [0,4]$

$\subseteq [-4,4]$.

$f([0,4]) = [0,16]$

$f^{-1}(f([0,4])) = f^{-1}([0,16]) = [-4,4] \supseteq [0,4]$.

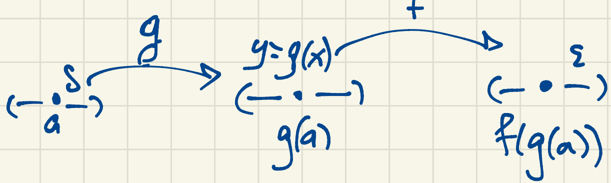
Theorem: Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous. Then $f \circ g : \mathbb{R} \to \mathbb{R}$ is continuous.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "g"] \arrow[rr, bend left, "f \circ g"] & \mathbb{R} \arrow[r, "f"] & \mathbb{R} \\ g^{-1}(f^{-1}(U)) & f^{-1}(U) & U \subseteq \mathbb{R} \end{tikzcd}$$

Proof (New proof) Let $U \subseteq \mathbb{R}$ be open. Then

$$(f \circ g)^{-1}(U) = \underbrace{g^{-1}(\underbrace{f^{-1}(U)}_{\text{open}})}_{\text{open}}$$ is open because $f^{-1}(U)$ is open.

Compare:
(Old proof) Let $a \in \mathbb{R}$, $\varepsilon > 0$. There exists $\delta_1 > 0$ such that $|f(y) - f(g(a))| < \varepsilon$ whenever $|y - g(a)| < \delta_1$.



Also there exists $\delta > 0$ such that $|g(x) - g(a)| < \delta_1$ whenever $|x - a| < \delta$.

So whenever $|x - a| < \delta$, we have $|f(g(x)) - f(g(a))| < \varepsilon$. $\square$