Analysis I (Math 3205)
Fall 2020

Book 3

Let $(a_n)_n$ be a sequence of real numbers. It is possible for such a sequence to have no limit point e.g. $a_n = n$. The sequence of positive integers has only isolated points. However, if (a_n) is bounded then it must have at least one limit point by the Bolzano-Weierstrass Theorem.

Eg. consider the sequence $(\sin n)_{n \in \mathbb{N}} = (\sin 1, \sin 2, \sin 3, \sin 4, \dots)$.

This sequence diverges. But the sequence is bounded (all terms lie in $[-1, 1]$)

So the sequence has a convergent subsequence. Thus there is at least one limit point. All limit points must lie in $[-1, 1]$.

$\sin 0 = 0.000\dots$

$\sin 1 = 0.841\dots$

$\sin 2 = 0.909\dots$

$\vdots$

$\sin 22 = -0.009$

$\sin 44 = 0.018$

$\sin 45 = 0.850$

$\sin 46 = 0.902$

$\pi \approx \frac{22}{7}$

$7\pi \approx 22$

$\sin 22 \approx \sin 7\pi = 0$

$\sin n \neq 0$ for any positive integer n because $\pi \notin \mathbb{Q}$. $\sin x = 0 \iff x = k\pi$ for some $k \in \mathbb{Z}$

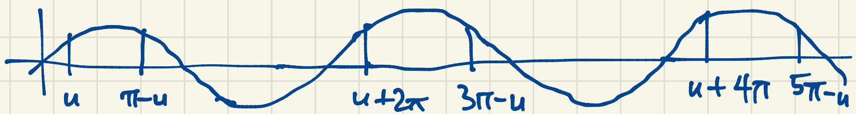
Also since $\pi \notin \mathbb{Q}$, the sequence $(\sin n)_n$ has no repeated terms and the limit points of $(\sin n)_n$ are all points of $[-1, 1]$.

$\left(\pi = \frac{n}{k} \in \mathbb{Q} \iff \sin n = 0 \iff n = k\pi\right.$ for some $k \in \mathbb{Z}$

If π is irrational then the sequence $(\sin n)_n$ has distinct terms (it never repeats).

Why? If $\sin u = \sin v$ then either $v - u = 2k\pi$ for some $k \in \mathbb{Z}$

or $v + u = (2k+1)\pi$ for some $k \in \mathbb{Z}$.



So if $\sin m = \sin n$ where $m \neq n$ are integers then either $m - n = 2k\pi$ with $0 \neq k \in \mathbb{Z}$ so $\pi = \frac{m-n}{2k} \in \mathbb{Q}$; or $m + n = (2k+1)\pi$ for some $k \in \mathbb{Z}$ so $\pi = \frac{m+n}{2k+1} \in \mathbb{Q}$ again contradicting $\pi \notin \mathbb{Q}$.

Let's prove $\pi \notin \mathbb{Q}$. Warm-up: prove $e \notin \mathbb{Q}$.

$0! = 1$

$1! = 1$

$2! = 1 \times 2 = 2$

$3! = 1 \times 2 \times 3 = 6$

$4! = 1 \times 2 \times 3 \times 4 = 24$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Recall: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

Suppose $e \in \mathbb{Q}$, say $e = \frac{a}{b}$ in lowest terms ($a, b \in \mathbb{N}$, $\gcd(a,b) = 1$).

$$\frac{a}{b} = e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

Suppose $e \in \mathbb{Q}$; say $e = \frac{a}{b}$ in lowest terms ($a, b \in \mathbb{N}$, $\gcd(a,b) = 1$).
Multiply both sides by $b! = 1 \times 2 \times 3 \times \cdots \times (b-1)b$.

$$b!\,\frac{a}{b} = \overbrace{(b-1)!\,a}^{\text{integer}} = b!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(b-1)!} + \frac{1}{b!} + \frac{1}{(b+1)!} + \cdots\right)$$

$$= \underbrace{b! + b! + \frac{b!}{2!} + \frac{b!}{3!} + \cdots + \underset{\substack{\| \\ b}}{\frac{b!}{(b-1)!}} + \underset{\substack{\| \\ 1}}{\frac{b!}{b!}}}_{\text{integers}} + \underbrace{\underset{\substack{\| \\ \frac{1}{b+1}}}{\frac{b!}{(b+1)!}} + \underset{\substack{\| \\ \frac{1}{(b+1)(b+2)}}}{\frac{b!}{(b+2)!}} + \cdots}_{\text{Not an integer } \cancel{\text{by comparison test}}}$$

The series $\frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \cdots$ converges by comparison with

$$\frac{1}{b+1} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \frac{1}{(b+1)^4} + \cdots = \frac{1/(b+1)}{1 - \frac{1}{b+1}} = \frac{1}{(b+1)-1} = \frac{1}{b} < 1.$$

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r} \quad (\text{for } |r| < 1).$$

This is a contradiction. So $e \notin \mathbb{Q}$.

$$(uv)' = u'v + uv'$$

$$(uv)'' = (u'v+uv')' = u''v + u'v' + u'v' + uv'' = u''v + 2u'v' + uv''$$

$$(uv)''' = (u''v+2u'v'+uv'')' = (u'''v + u''v') + 2(u''v' + u'v'') + (u'v'' + uv''')$$

$$= u'''v + 3u''v' + 3u'v'' + uv'''$$

$$(u+v)^1 = u+v$$

$$(u+v)^2 = u^2 + 2uv + v^2$$

$$(u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$$

$$(u+v)^n = \sum_{k=0}^{n} \binom{n}{k} u^k v^{n-k}$$

(Binomial Theorem)

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \in \mathbb{Z}$$

"binomial coefficients" are the entries in Pascal's Triangle

Leibniz' Formula

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}$$

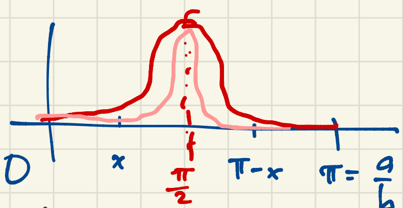
Theorem $\pi \notin \mathbb{Q}$.

Proof Suppose $\pi = \frac{a}{b}$ in lowest terms (i.e. $a, b \in \mathbb{N}$, $\gcd(a,b)=1$). We look for a contradiction. Consider the function $f(x) = \frac{1}{n!} x^n (a-bx)^n$ where $n \in \mathbb{N}$ will be chosen later. Note: $f(x) = u(x)v(x)$ where $u(x) = \frac{1}{n!} x^n$, $v(x) = (a-bx)^n$.

Lemma For every $k \geq 0$, $f^{(k)}(0) = (-1)^k f^{(k)}(\pi) \in \mathbb{Z}$.

Proof $f(\pi - x) = f(\frac{a}{b} - x) = \frac{1}{n!}(\frac{a}{b} - x)^n (a - b(\frac{a}{b} - x))^n = \frac{1}{n!}(\frac{a}{b} - x)^n (a - (a - bx))^n$

$$= \frac{1}{n!}\left(\frac{a}{b} - x\right)^n (bx)^n = \frac{1}{n!}\left(\frac{a}{b} - x\right)^n b^n x^n = \frac{1}{n!}\left(\left(\frac{a}{b} - x\right)b\right)^n x^n = \frac{1}{n!}(a-bx)^n x^n = f(x).$$



$f(x) = \frac{1}{n!}(ax - bx^2)^n$

$$f(\pi - x) = f(x)$$
$$-f'(\pi - x) = f'(x)$$
$$f''(\pi - x) = f''(x)$$
$$-f'''(\pi - x) = f'''(x)$$

$$f^{(k)}(x) = (-1)^k f^{(k)}(\pi - x)$$
$$f^{(k)}(0) = (-1)^k f^{(k)}(\pi)$$

We must show this $\in \mathbb{Z}$.

$f(x) = u(x)v(x)$, $\quad u(x) = \frac{1}{n!} x^n$

$u'(x) = \frac{1}{n!} \cdot n x^{n-1} = \frac{1}{(n-1)!} x^{n-1}$

$u''(x) = \frac{1}{(n-2)!} x^{n-2}$

$\ldots \quad u^{(n-2)}(x) = \frac{1}{2!} x^2 = \frac{1}{2} x^2$

$u^{(n-1)}(x) = x \qquad u^{(n+1)}(x) = 0$

$u^{(n)}(x) = 1 \qquad u^{(n+2)}(x) = 0$ etc.

$$u^{(k)}(0) = \begin{cases} 0 & \text{if } k \neq n; \\ 1 & \text{if } k = n. \end{cases} \in \mathbb{Z}$$

Recall: $f(x) = u(x)v(x)$, $\quad u(x) = \frac{1}{n!}x^n$, $\quad v(x) = \underbrace{(a-bx)^n}$ $\in \mathbb{Z}[x]$

i.e. a polynomial in x with integer coefficients

$$f^{(k)}(x) = \sum_{r=0}^{k} \binom{k}{r} u^{(r)}(x)\, v^{(k-r)}(x)$$

$$f^{(k)}(0) = \sum_{r=0}^{k} \underbrace{\binom{k}{r}}_{\text{integers}} \underbrace{u^{(r)}(0)}_{\text{integers}}\, \underbrace{v^{(k-r)}(0)}_{\text{integers}} \in \mathbb{Z}.$$

This proves the Lemma.

Note: $f(x)$ is a poly. in x of degree $2n$.

Return to the Theorem.

$$F''(x) = f''(x) - f^{(4)}(x) + f^{(6)}(x) - \cdots + (-1)^{n-1} f^{(2n)}(x)$$

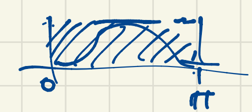
Consider $F(x) = f(x) - f''(x) + f^{(4)}(x) - f^{(6)}(x) + \cdots + (-1)^n f^{(2n)}(x)$

$$\frac{d}{dx}\left[F'(x)\sin x - F(x)\cos x\right] = F''(x)\sin x + \cancel{F'(x)\cos x} - \left(\cancel{F'(x)\cos x} - F(x)\sin x\right)$$

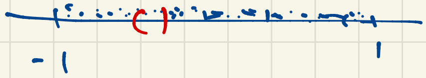
$$= \left[F''(x) + F(x)\right]\sin x = f(x)\sin x$$

$$\int_0^\pi f(x)\sin x\,dx = \left[F'(x)\sin x - F(x)\cos x\right]_0^\pi = F(\pi) - F(0) = F(0) - F\left(\tfrac{a}{b}\right) \in \mathbb{Z}$$ by the Lemma

$$0 < \int_0^\pi f(x)\sin x\,dx < \pi \cdot f\left(\tfrac{\pi}{2}\right) = \frac{\pi}{n!}\left(\frac{\pi}{2}\right)^{2n} \to 0 \text{ as } n \to \infty.$$



$f(x) = \frac{1}{n!}(ax - bx^2)^n$ is maximized at $x = \frac{\pi}{2} = \frac{a}{2b}$ on $[0, \pi]$. For n sufficiently large the integral is in $(0,1)$, it can't be an integer. □



Topology

Recall: If $A \subseteq \mathbb{R}^n$, a limit point of A $b \in \mathbb{R}^n$ is a point such that for all $\varepsilon > 0$, there exists $a \in A$ satisfying $0 < |a - b| < \varepsilon$.

The *derived* set of A is A' = the set of all limit points of A. Note: Limit points of A can belong to A but they don't have to.

eg. $[0,1)' = [0,1]$ $\qquad \overline{[0,1)} = [0,1]$

$\mathbb{Z}' = \emptyset$ $\qquad \overline{\mathbb{Z}} = \mathbb{Z}$

$\mathbb{Q}' = \mathbb{R}$ $\qquad \overline{\mathbb{Q}} = \mathbb{R}$

The *closure* of A is $\overline{A} = A \cup A'$.

Note: $\overline{\overline{A}} = \overline{A}$.

An <u>open</u> set in $\mathbb{R}^n$ is a union of open balls.

In $\mathbb{R}$, an open ball $B_r(a) = \{x \in \mathbb{R} : |x-a| < r\}$ of radius r centered at $a \in \mathbb{R}$ is the same thing as an open interval $(a-r, a+r)$.

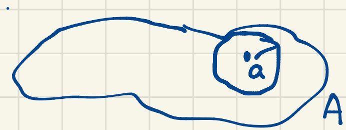
Every open interval (a,b) is an open set. $(a,b) = B_{\frac{b-a}{2}}\left(\frac{a+b}{2}\right)$

Also $(a,\infty) = \bigcup_{c \geq a} (c, c+1)$ is open.

$[0,1]$ is not open.

$[0,1)$ is not open. Proof: If $[0,1) = \bigcup_{i \in I} (a_i, b_i)$ for some collection of open intervals $\{(a_i, b_i) : i \in I\}$ then $0 \in (a_i, b_i)$ for some $i \in I$. Every such interval also contains some negative numbers, a contradiction.

<u>Alternatively</u>, a subset $A \subseteq \mathbb{R}^n$ is open if every $a \in A$ lies inside a ball $B_\delta(a) \subseteq A$ for some $\delta > 0$.



A set $A \subseteq \mathbb{R}$ is <u>closed</u> if it contains all its limit points (i.e. $A' \subseteq A$ i.e. $\bar{A} = A$) eg. $[a,b]$ is closed. $[a,b]' = [a,b]$.

$\overline{[a,b]} = [a,b] \cup [a,b]' = [a,b]$.

$[a,\infty)$ is closed. $[a,\infty)' = [a,\infty)$, $\overline{[a,\infty)} = [a,\infty) \cup [a,\infty)' = [a,\infty)$

$\bar{A}$ is the smallest closed set containing A.

$\mathbb{Z}$ is closed.

$\mathbb{Q}$ is not closed. $\bar{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}' = \mathbb{Q} \cup \mathbb{R} = \mathbb{R}$.

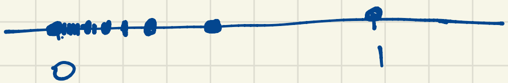
$\mathbb{Q}$ is not open. ($0 \in \mathbb{Q}$ is not covered by any $B_\delta(0) = (-\delta,\delta)$ for $\delta > 0$ inside $\mathbb{Q}$.)

Let $A \subseteq \mathbb{R}^n$. Then A is open iff its complement $\mathbb{R}^n - A$ is closed.

eg. $\mathbb{Z}$ is closed. $\mathbb{R} - \mathbb{Z} = \bigcup_{n \in \mathbb{Z}} (n, n+1) = \cdots \cup (-2,-1) \cup (-1,0) \cup (0,1) \cup (1,2) \cup (2,3) \cup \cdots$ is open.

eg. $A = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$ is neither open nor closed.

$A' = \{0\}$. $\bar{A} = A \cup A' = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$ is closed.



$\bar{A}$

Its complement is open: $\mathbb{R} - \bar{A} = (-\infty, 0) \cup (1, \infty) \cup \left(\bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right)\right)$

$= (-\infty, 0) \cup (1, \infty) \cup (\frac{1}{2}, 1) \cup (\frac{1}{3}, \frac{1}{2}) \cup (\frac{1}{4}, \frac{1}{3}) \cup (\frac{1}{5}, \frac{1}{4}) \cup \dots$

Can a set be both open and closed?

$\emptyset$ is both open and closed (i.e. clopen)

$\mathbb{R}$ is clopen.

$\emptyset$ and $\mathbb{R}$ are the only clopen sets in $\mathbb{R}$. This is an important theorem which forms the basis for the Intermediate Value Theorem. The proof uses the completeness of $\mathbb{R}$.

Let X be a set. (eg. $\mathbb{R}$ or $\mathbb{R}^n$). A topology on X is a collection $\mathcal{T}$ of subsets of X (script T) called the open sets, satisfying:

- $\emptyset, X \in \mathcal{T}$
- whenever $A, B \in \mathcal{T}$, we have $A \cap B \in \mathcal{T}$.
- whenever $\{A_i : i \in I\} \subseteq \mathcal{T}$, $\bigcup_{i \in I} A_i \in \mathcal{T}$.

$\emptyset, X$ are open.
Unions of open sets are open –
Intersections of finitely many open sets are open.

eg. $\bigcap_{n \in \mathbb{N}} (0, \frac{n+1}{n}) = (0,2) \cap (0, \frac{3}{2}) \cap (0, \frac{4}{3}) \cap (0, \frac{5}{4}) \cap (0, \frac{6}{5}) \cap \cdots = (0, 1]$ is not open.

$\emptyset, X$ are closed.
Intersections of closed sets are closed.
Unions of finitely many closed sets are closed.

eg. $\bigcup_{0<\delta<1} [0, \delta] = [0, 1)$ is not closed.

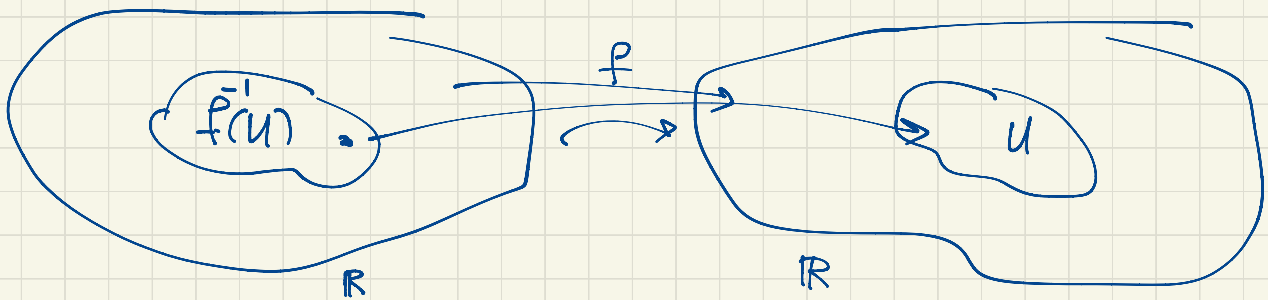
Eg. The Cantor Set is closed.

$C = [0,1] \cap ([0, \frac{1}{3}] \cup [\frac{2}{3}, 1]) \cap ([0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]) \cap \cdots$ is closed since it is an intersection of closed sets.

Recall: $f:\mathbb{R}\to\mathbb{R}$ is continuous if for all $a\in\mathbb{R}$ and $\varepsilon>0$, there exists $\delta>0$ such that
$|f(x)-f(a)|<\varepsilon$ whenever $|x-a|<\delta$.

The following statement is equivalent as a definition of continuity:
For every open $U\subseteq\mathbb{R}$, $f^{-1}(U)$ is also open in $\mathbb{R}$.
("The preimage of every open set is open").

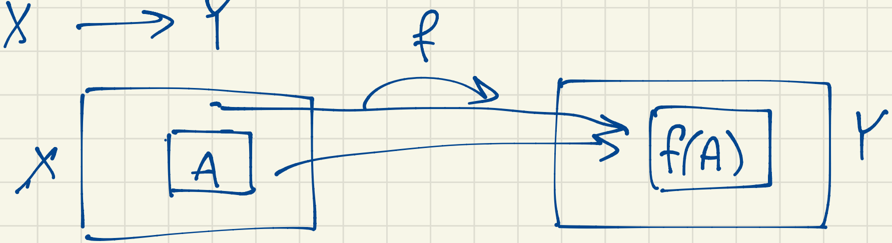
Note: We are not assuming f is one-to-one. f may not have an inverse function!



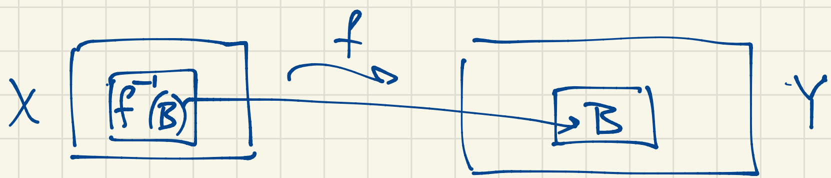
For every $U\subseteq\mathbb{R}$, define $f^{-1}(U)=\{x\in\mathbb{R} : f(x)\in U\}$
(the preimage of U under f).

Compare: $f(U) = \{ f(u) : u \in U \}$ (the image of U under f).

If $f : X \longrightarrow Y$



$f^{-1}(f(A)) \supseteq A$ for all $A \subseteq X$.



$f(f^{-1}(B)) \subseteq B$

eg. $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$.

$f^{-1}([-4,4]) = [-2,2]$

$f(f^{-1}([-4,4])) = f([-2,2]) = [0,4]$
$\subseteq [-4,4]$.

$f([0,4]) = [0,16]$

$f^{-1}(f([0,4])) = f^{-1}([0,16]) = [-4,4] \supseteq [0,4]$.

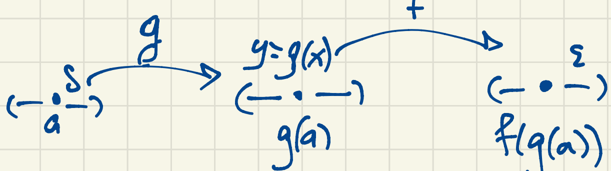
Theorem: Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous. Then $f \circ g : \mathbb{R} \to \mathbb{R}$ is continuous.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "g"] \arrow[rr, bend left, "f \circ g"] & \mathbb{R} \arrow[r, "f"] & \mathbb{R} \\ g^{-1}(f^{-1}(U)) & f^{-1}(U) & U \arrow[u, phantom, "\subseteq" rotate=90] \end{tikzcd}$$

Proof (New proof) Let $U \subseteq \mathbb{R}$ be open. Then

$$(f \circ g)^{-1}(U) = \underbrace{g^{-1}(\underbrace{f^{-1}(U)}_{\text{open}})}_{\text{open}}$$ is open because $f^{-1}(U)$ is open.

Compare:
(Old proof) Let $a \in \mathbb{R}$, $\varepsilon > 0$. There exists $\delta_1 > 0$ such that $|f(y) - f(g(a))| < \varepsilon$ whenever $|y - g(a)| < \delta_1$.

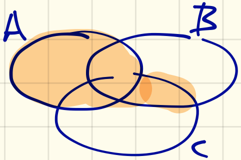


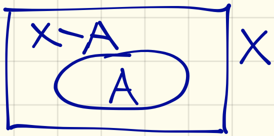
Also there exists $\delta > 0$ such that $|g(x) - g(a)| < \delta_1$ whenever $|x - a| < \delta$.
So whenever $|x - a| < \delta$, we have $|f(g(x)) - f(g(a))| < \varepsilon$. $\square$

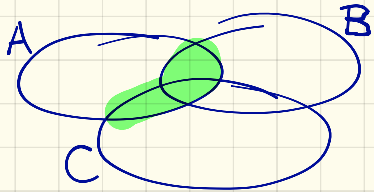
Distributive Laws

$A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

$A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$







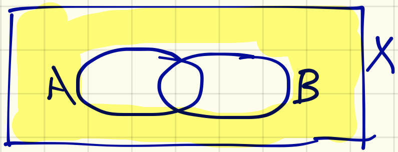
$X-A = \{x\in X \quad x\notin A\}$

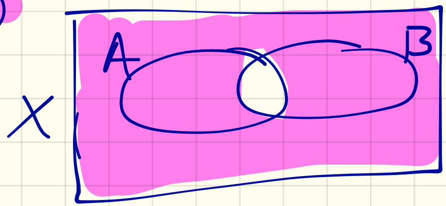
De Morgan's Laws

If $A,B\subseteq X$,

$\longrightarrow$ $X-(A\cup B) = (X-A)\cap(X-B)$

$X-(A\cap B) = (X-A)\cup(X-B)$





$\emptyset, X$ are open

$\{A_\iota \quad \iota\in I\}$ open sets $\Longrightarrow$ $\bigcup_{\iota\in I} A_\iota$ open

$A_1, \quad , A_n$ open $\Longrightarrow$ $A_1\cap A_2\cap \quad \cap A_n$ open

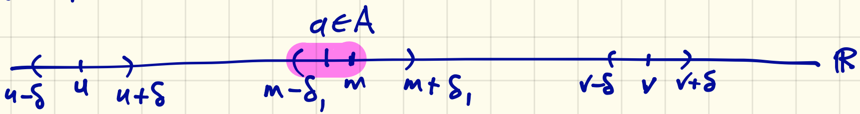
$\emptyset, X$ closed $(X, \emptyset$ open$)$

$U_\iota = X-K_\iota$ open

$\{K_\iota \quad \iota\in I\}$ closed sets $\Longrightarrow$ $\bigcap_{\iota\in I} K_\iota = \bigcap_{\iota\in I}(X-U_\iota) = X-\bigcup_{\iota\in I} U_\iota$ is closed

Theorem The only clopen sets in $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$

Proof Suppose $U \neq \emptyset, \mathbb{R}$ is clopen ie $\mathbb{R} = U \sqcup V$ where U, V are disjoint nonempty open sets. Let $u \in U$, $v \in V$. Without loss of generality, $u < v$



There exists $\delta > 0$ such that $(u-\delta, u+\delta) \subseteq U$ (since U is open) and $(v-\delta, v+\delta) \subseteq V$

Let A be the set of all $a \in [u,v]$ such that $[u,a) \subseteq U$. Clearly $u+\delta \in A$, $v \notin A$, $[u, u+\delta) \subseteq A \subseteq [u, v-\delta]$. Since A is a bounded nonempty subset of $\mathbb{R}$ it has a least upper bound $m = \sup A$ ie $[u,m) \subseteq U$ but $[u,a) \not\subseteq U$ for $a > m$

$u+\delta \leq m \leq v-\delta$ Note either $m \in U$ or $m \in V$

If $m \in U$ then there exists $\delta_1 > 0$ such that $(m-\delta_1, m+\delta_1) \subseteq U$ (we make sure $\delta_1 < \delta$ so that this interval stays inside $[u,v]$)

Since $m = \sup A$, there exists $a \in A$, $m-\delta_1 \leq a \leq m$. Then $[u,a) \subseteq U$, $(m-\delta_1, m+\delta_1) \subseteq U$ so their union $[u, m+\delta_1) \subseteq U$ so $m+\delta_1 \in A$, $m+\delta_1 > m$ contradicting $m = \sup A$

If $m \in V$ we get a similar contradiction $\square$

Intermediate Value Theorem If $f: \mathbb{R} \to \mathbb{R}$ is continuous taking some positive value and some negative value, then $f(c) = 0$ for some $c \in \mathbb{R}$

(Remark: later we will consider functions $f: [a,b] \to \mathbb{R}$ and even more general domains than this)

Proof: Suppose $\underbrace{f(\mathbb{R})}_{\{f(x):\, x\in\mathbb{R}\}} \subseteq (-\infty,0) \cup (0,\infty)$. We must find a contradiction.

Then $\mathbb{R} = \underbrace{f^{-1}((-\infty,0))}_{\{x\in\mathbb{R}:\, f(x)<0\}} \sqcup \underbrace{f^{-1}((0,\infty))}_{\{x\in\mathbb{R}:\, f(x)>0\}}$ is a disjoint union of two nonempty open sets, a contradiction. $\square$

When we say the only clopen sets in $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$, this is saying $\mathbb{R}$ is connected.

$\mathbb{Q}$ is not connected

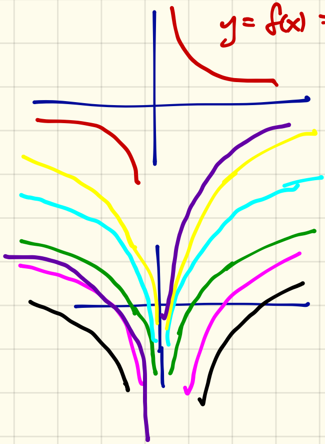
$$\mathbb{Q} = \{x\in\mathbb{Q} : x<\sqrt{2}\} \sqcup \{x\in\mathbb{Q} : x>\sqrt{2}\}$$

nonempty open in $\mathbb{Q}$

Recall: An antiderivative for f is a function F such that $F' = f$

What are the possible antiderivatives of $f(x) = \frac{1}{x}$?

One antiderivative is $\ln|x| = F(x)$



An antiderivative for $f(x) = \frac{1}{x}$ is
$F(x) = \ln|x|$

Another antiderivative is
$F(x) + 1$

A more general antiderivative is
$F(x) + C = \ln|x| + C$
where $C \in \mathbb{R}$ is any constant

Are there others?

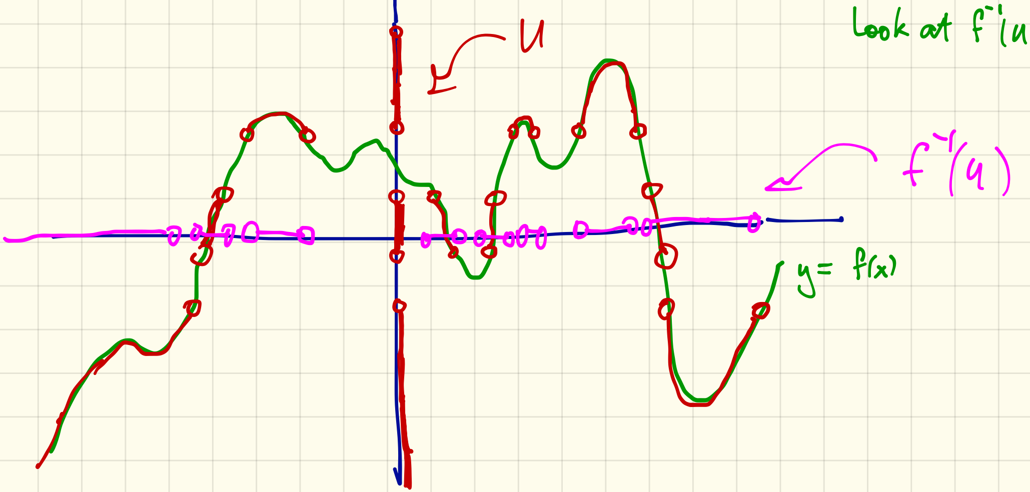
Why do we need more than one arbitrary constant to express the antiderivative of f?
Because the domain of f in general won't be connected

The general antiderivative (i.e. the most general antiderivative) for f is

$$\begin{cases} \ln x + C_1 & \text{for } x > 0 \\ \ln|x| + C_2 & \text{for } x < 0 \end{cases}$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary real constants

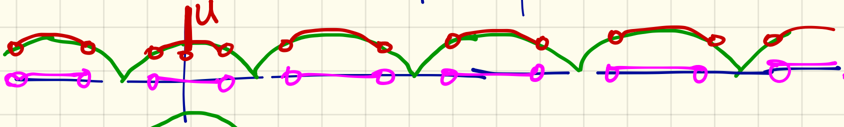
Look at $f^{-1}(U)$ for a typical open set U



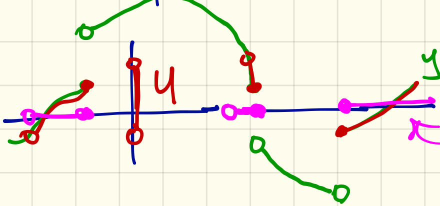
$= \{x \in \mathbb{R} \mid |\cos x| > 0.8\}$

$= \{x \mid f(x) \in (0.8, \infty)\}$

Eg $\{x \in \mathbb{R} \mid |\cos x| > 0.8\}$ is an open set in $\mathbb{R}$ since it is $f^{-1}((0.8,\infty))$ where $f(x) = |\cos x|$



$f^{-1}((0.8,\infty))$ is open



$y = f(x)$ is not continuous

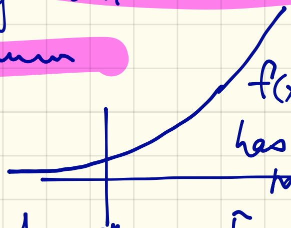
$f^{-1}(U)$ is not open

Big theorem from Calculus I

Every continuous function $f\ [a,b] \to \mathbb{R}$ has a maximum and a minimum

closed ~~bounded~~ interval

Eg



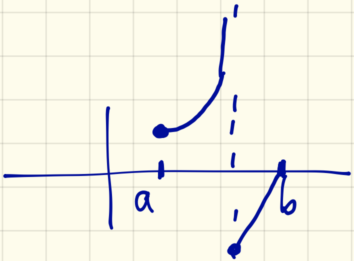
$f(x) = e^x$, $x \in \mathbb{R}$ has no maximum and no minimum

$[0,\infty)$ is a closed unbounded interval

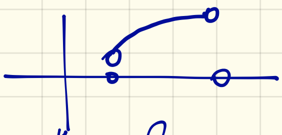
f is continuous

f is not bounded above

f is bounded below, with 0 as a lower bound (0 is the greatest lower bound, ie the infimum of f) 0 is not a value of f so it's certainly not a minimum value



Here is a discontinuous function defined on $[a,b]$ with a minimum but no maximum

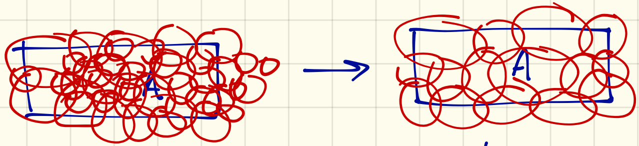


Here is a continuous function ~~on~~ an open interval with no maximum and no minimum

The relevance of $[a,b]$ is that this is a <u>compact</u> set

Let's say what it means for a set $A \subseteq \mathbb{R}$ (or $\mathbb{R}^n$) to be compact

An open cover of A is a collection of open sets $\{U_i \quad i \in I\}$ covering A, ie
$$A \subseteq \bigcup_{i \in I} U_i$$



It may often happen that a given open cover has a smaller subcover ie $\{U_i \quad i \in I'\}$, $I' \subseteq I$ such that $A \subseteq \bigcup_{i \in I'} U_i$ Such a subcollection is called an open subcover

We say A is compact if every open cover of A has a finite subcover

$\mathbb{R}$ is not compact It has an open cover consisting of all open intervals $(a, a+1)$ of length 1 This has no finite subcover.

$\{2, 5, 9\} \subset \mathbb{R}$ is compact.

Heine-Borel Theorem $[0, 1]$ is compact

Theorem Given $A \subseteq \mathbb{R}$, the following conditions are equivalent:

(i) A is compact (ie. every open cover of A has a finite subcover)

(ii) A is closed and bounded

(iii) A is sequentially compact ie. every sequence in A has a subsequence converging in A. (This means converging to a point of A).

The equivalence (i) $\Leftrightarrow$ (ii) is by the Heine-Borel Theorem. The equivalence (i) $\Leftrightarrow$ (iii) is another theorem.

Advice for doing mathematics:

- When you encounter a new topic / definition / theorem, put it to the test using examples.
- Make sure you learn the examples, not just the theorems.
- Don't start by paraphrasing.
- When learning a new topic, trust that the author / book / content is useful, beautiful, valid, coherent, etc.

Eg. $\mathbb{Z}$ is not compact. $\{(a,a+1) : a \in \mathbb{R}\}$ is an open cover of $\mathbb{Z}$. Every finite subcollection $\{(a_i, a_i+1) : i = 1,2,\dots,n\}$ is bounded and

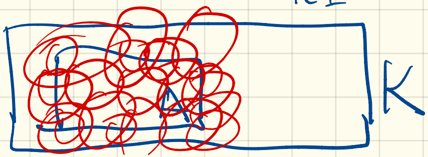
$$\bigcup_{i=1}^{n} (a_i, a_i+1) \subseteq (r,s) \quad \text{where } r = \min\{a_1,\dots,a_n\},$$
$$s = \max\{a_1,\dots,a_n\}+1.$$

which is bounded so it doesn't cover $\mathbb{Z}$. Note that $\mathbb{Z}$ is closed (it has no limit points) but not bounded.

Eg. The Cantor Set is compact. It is bounded (a subset of $[0,1]$) and it is closed.

<u>Theorem</u> Every closed subset of a compact set is compact.

<u>Proof</u> Let K be compact and let $A \subseteq K$ be closed. So $A' = \mathbb{R} - A$ is open. Let $\{U_i : i \in I\}$ be an open cover of A. (Thus $U_i \subseteq \mathbb{R}$ is open for all $i \in I$; and $A \subseteq \bigcup_{i \in I} U_i$. Then $\underbrace{\{U_i : i \in I\}}_{\text{covers } A} \cup \underbrace{\{A'\}}_{\text{covers } K-A}$ is an open cover of K.



Since K is compact, this open cover has a finite subcover $\{U_1, U_2, \dots, U_n, A'\}$ so $\{U_1, \dots, U_n\}$ covers A. $\square$

Theorem Every nonempty compact set $K \subseteq \mathbb{R}$ has a maximum and a minimum.

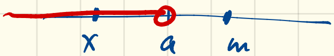
Think: $(0,1] \subset \mathbb{R}$ is not compact. It has no maximum or minimum.

Proof of the Theorem: Let $K \subseteq \mathbb{R}$ be a nonempty compact set.

So K is bounded. ($K \subseteq \bigcup_{a\in\mathbb{R}} (-\infty,a) = \mathbb{R} \implies K \subseteq (-\infty,a_1) \cup \cdots \cup (-\infty,a_n)$ for some $a_1,\ldots,a_n \in \mathbb{R}$ so $K \subseteq (-\infty,a)$ where $a = \max\{a_1,\ldots,a_n\}$ so a is an upper bound for K.) So K has a least upper bound $m = \sup K$. I need to show $m \in K$ (in which case it is the maximum element of K). Continued on Tuesday ...

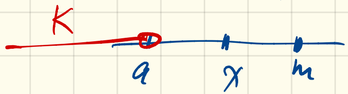
If $m \notin K$, $K \subseteq \bigcup_{a<m} (-\infty,a)$.

(For every $x \in K$, $x \le m$ so $x < m$. Pick $a \in (x,m)$ so $x \in (-\infty,a)$ where $a<m$, so $x \in \bigcup_{a<m}(-\infty,a)$.)



Since K is compact,

$K \subseteq (-\infty,a_1) \cup (-\infty,a_2) \cup \cdots \cup (-\infty,a_n) = (-\infty,a)$ where $a = \max\{a_1,a_2,\ldots,a_n\} < m$ for some $a_1,a_2,\ldots,a_n < m$. Pick $x \in (a,m)$. So $x \notin K$. In fact x is an upper bound for $K \subseteq (-\infty,a)$



Contradicting $x < m$ where m is the least upper bound for K.

So $m = \sup K \in K$ which must be the maximum element of K.

Theorem Let $f: X \to Y$ be continuous. I will assume f is onto i.e. surjective (so Y is the image of f i.e. $f(X) = Y$.)

Remark: Y is usually called the range. But be careful: some books say "range" as a synonym for image.

(i) If X is connected, then Y is connected.

(ii) If X is compact, then so is Y.

Proof (i) Suppose Y is disconnected; then we must show X is disconnected. (This is the contrapositive.) If $Y = U \sqcup V$ where U and V are open nonempty, then $X = f^{-1}(U) \sqcup f^{-1}(V)$ where $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint nonempty open sets.

(ii) Let $\{U_i : i \in I\}$ be an open cover of Y. So $U_i \subseteq Y$ is open for all i and $Y \subseteq \bigcup_{i \in I} U_i$. Then $\{f^{-1}(U_i) : i \in I\}$ is an open cover of X.

$f(A \cup B) = f(A) \cup f(B)$

$X = f^{-1}(Y) \subseteq \bigcup_{i \in I} f^{-1}(U_i)$ $\quad$ ($A \subseteq B \Rightarrow f^{-1}(A) \subseteq f^{-1}(B)$)

Since X is compact, $X \subseteq f^{-1}(U_{i_1}) \cup f^{-1}(U_{i_2}) \cup \dots \cup f^{-1}(U_{i_n})$ for some $i_1, \dots, i_n \in I$. So $Y \subseteq U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_n}$.

Intermediate Value Theorem If $f : [a,b] \to \mathbb{R}$ is continuous with $f(a) < 0 < f(b)$, then $f(c) = 0$ for some $c \in (a,b)$.

Proof $f([a,b])$ is a connected subset of $\mathbb{R}$. (since it is the image of an interval $[a,b]$ which is connected). See the video on Topology. If $0 \notin f([a,b])$ then $f([a,b]) = U \sqcup V$ where $U = f([a,b]) \cap (-\infty, 0)$, $V = f([a,b]) \cap (0,\infty)$.

U, V are nonempty since $f(a) \in U$, $f(b) \in V$. They are open subsets of the image. So we have a continuous function f taking a connected domain $[a,b]$ to a disconnected image $f([a,b])$, contradiction. $\square$

ASIDE

Subspace topology: If $A \subseteq \mathbb{R}$ then A inherits a topology from $\mathbb{R}$: open sets in A are $O \cap A$ where $O \subseteq \mathbb{R}$ is open.

eg. $\mathbb{Q} \subseteq \mathbb{R}$ is a subspace whose open sets look like $O \cap \mathbb{Q}$ where $O \subseteq \mathbb{R}$ is open.

$\mathbb{Q} = O_1 \sqcup O_2$, $O_1 = \mathbb{Q} \cap (-\infty, \sqrt{2})$ is open in $\mathbb{Q}$ (not in $\mathbb{R}$ though)

$O_2 = \mathbb{Q} \cap (\sqrt{2}, \infty)$ is open in $\mathbb{Q}$

So $\mathbb{Q}$ is disconnected.

Theorem If $f: [a,b] \to \mathbb{R}$ is continuous then f has a maximum and a minimum. (There exists $c \in [a,b]$ such that $f(x) \le f(c)$ for all $x \in [a,b]$. Similarly for minimum.) c is a maximum point; $f(c)$ is the maximum value.

Proof $[a,b]$ is compact (by the Heine-Borel Theorem) so $f([a,b])$ is compact, hence closed and bounded. Also $[a,b]$ is connected so $f([a,b])$ is connected. So $f([a,b])$ is an interval.
So $f([a,b]) = [m_1, m_2]$ for some $m_1, m_2 \in \mathbb{R}$.
Then m_1 is the minimum value of f and m_2 is the maximum value of f.
The values of f are all the numbers between m_1 and m_2, inclusive. $\square$

More generally if $K \subseteq \mathbb{R}$ is compact then every continuous function $f: K \to \mathbb{R}$ has a maximum and a minimum.

Let $A \subseteq \mathbb{R}$. Then A is connected iff A is an interval, i.e.

(a,b), $(a,b]$, $[a,b)$, $[a,b]$, $[a,\infty)$, (a,∞), $(-\infty,b)$, $(-\infty,b]$, $(-\infty,\infty)=\mathbb{R}$

Consider the sequence of functions $f_n(x) = x^n$, $0 \le x \le 1$. These are $(n=1,2,3,\dots)$ continuous functions.

Let $f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & \text{for } 0 \le x < 1; \\ 1, & \text{if } x = 1. \end{cases}$

Note $f_n \to f$ but f is discontinuous whereas f_n is continuous.

We have taken the limit $f_n(x) \to f(x)$ pointwise (for each $x \in [0,1]$)

The convergence is not uniform.

$\lim_{n\to\infty} f_n(x) = f(x)$ says: For all $x \in [0,1]$ and $\varepsilon > 0$, there exists N such that

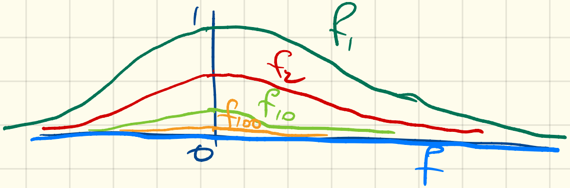
$$|f_n(x) - f(x)| < \varepsilon \quad \text{whenever } n > N.$$

We take $f_n(x)$ as a sequence of numbers for $n=1,2,3,\dots$ where $x \in [0,1]$ is fixed.

Note that $N = N(\varepsilon, x)$. The value of N will need to be larger if ε is taken as smaller; but also if x is taken as closer to 1.

We say $f_n(x) \to f(x)$ converges uniformly for $x \in A$ if N can be chosen independently of the choice of $x \in A$ (i.e. N depends only on ε).

Eg. $f_n(x) = \dfrac{1}{n+x^2}$, for $x \in \mathbb{R}$, $n \geq 1$.



$f_n(x) \longrightarrow f(x) = 0$ uniformly on $\mathbb{R}$

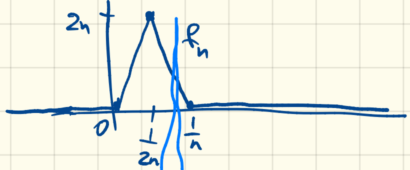
For all $\varepsilon > 0$ there exists N such that $|f_n(x) - f(x)| < \varepsilon$
for all $n > N$ and all $x \in \mathbb{R}$.

Here $N = N(\varepsilon)$ is independent of the choice of $x \in \mathbb{R}$ (it is chosen uniformly for the entire domain); it only depends on ε.

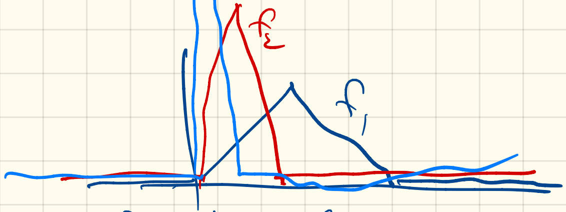
Here $N = N(\varepsilon) = \dfrac{1}{\varepsilon}$. If $n > \dfrac{1}{\varepsilon}$ then $|f_n(x) - \underbrace{f(x)}_{0}| = \dfrac{1}{n+x^2} \leq \dfrac{1}{n} < \varepsilon$

$\boxed{\varepsilon > \frac{1}{n}}$

Eg. $f_n(x) = \begin{cases} 4n^2 x, & \text{if } 0 \le x \le \frac{1}{2n} \\ 4n^2\left(\frac{1}{n} - x\right), & \text{if } \frac{1}{2n} \le x \le \frac{1}{n} \\ 0, & \text{otherwise.} \end{cases}$



$\lim_{n\to\infty} f_n(x) = f(x) = 0$ for all x.



The convergence is not uniform

Here the limit of the continuous functions f_n is a continuous function f. But there is another problem:

$$\int_{-\infty}^{\infty} f_n(x)\,dx = \frac{1}{2}\cdot\frac{1}{n}\cdot 2n = 1. \text{ whereas } \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} 0\,dx = 0.$$

$$\underbrace{\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\,dx}_{\lim_{n\to\infty} 1 = 1} \neq \underbrace{\int_{-\infty}^{\infty} \lim_{n\to\infty} f(x)\,dx}_{\int_{-\infty}^{\infty} 0\,dx = 0.}$$

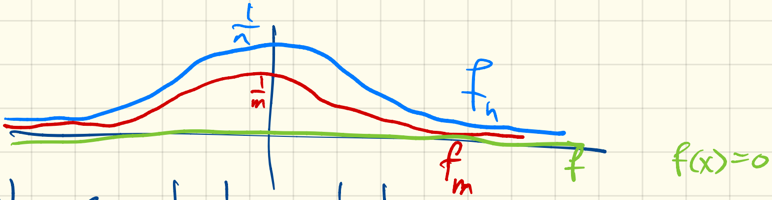
The failure of $\lim \int f_n$ to equal $\int \lim f_n$ is due to the fact that our convergence is not uniform.

Another way to view the distinction between pointwise and uniform convergence:
Define the distance between two functions $f, g: A \longrightarrow \mathbb{R}$ to be

$$d(f,g) = \| f-g \| \quad \text{where} \quad \| f \| = \sup_A |f| = \sup \{ |f(a)| : a \in A \}.$$

Sometimes written as $\| f \|_\infty$. (Remark: It is usually preferable to ignore sets of measure zero in the domain.)

eg. $f_n(x) = \dfrac{1}{n+x^2}$, $n \in \mathbb{N}$



$$\| f_m - f_n \| = \sup_{x \in \mathbb{R}} |f_m(x) - f_n(x)| = \sup_{x \in \mathbb{R}} \left| \frac{1}{m+x^2} - \frac{1}{n+x^2} \right| = \frac{1}{m} - \frac{1}{n} \quad \text{if } m < n.$$

$$|g(x)| = f_m(x) - f_n(x) = \frac{1}{m+x^2} - \frac{1}{n+x^2} \qquad \text{if } m < n$$

$$g'(x) = \frac{2x(m-n)(2x^2+m+n)}{(x^2+m)^2(x^2+n)^2}$$

$g'(x) < 0$ for $x > 0$ i.e. $g(x)$ is decreasing on $(0,\infty)$

$g'(x) > 0$ for $x < 0$ is $g(x)$ is increasing on $(-\infty, 0)$.

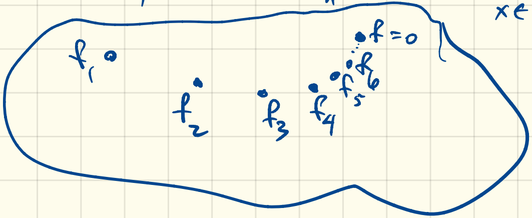
So $g(x)$ has a unique maximum $g(0) = \frac{1}{m} - \frac{1}{n}$.

$$\boxed{\| f_m - f_n \| = \left| \frac{1}{m} - \frac{1}{n} \right|}$$

$f_n(x) = \frac{1}{n+x^2} \longrightarrow f(x) = 0$ pointwise but also $f_n \to f$ in the sup-norm i.e.

$d(f_n, f) \to 0$ as $n \to \infty$.

$$d(f_n, f) = \| f_n - f \| = \sup_{x \in \mathbb{R}} \left| \frac{1}{n+x^2} - 0 \right| = \frac{1}{n}$$



(STRONG CONVERGENCE)

To say $f_n \to f$ in norm is much stronger than saying $f_n(x) \to f(x)$ pointwise. $f_n \to f$ in norm implies uniform convergence.

(WEAK CONVERGENCE)

<u>Theorem</u> If $f_n \to f$ in norm then $f_n(x) \to f(x)$ pointwise.

(But not conversely.)

<u>Proof</u> Let $x \in A$ (the domain) and let $\varepsilon > 0$. Since $f_n \to f$ in (sup) norm, there exists $N = N(\varepsilon)$ (i.e. independent of x, i.e. uniformly for all $x \in A$) such that $\sup_{a \in A} |f_n(a) - f(a)| < \varepsilon$ for all $n > N$. Then

$$|f_n(x) - f(x)| \leq \sup_{a \in A} |f_n(a) - f(a)| < \varepsilon \quad \text{for all } n > N.$$

Thus $\lim_{n \to \infty} f_n(x) = f(x)$. $\square$

Why does the converse fail?

Consider $f_n(x) = x^n, \quad 0 \le x \le 1$.

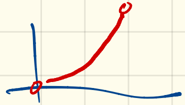
$f_n(x) \longrightarrow f(x)$ pointwise on $[0,1]$ where $f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1; \\ 1, & \text{if } x = 1. \end{cases}$

Does $f_n \to f$ in norm? No:

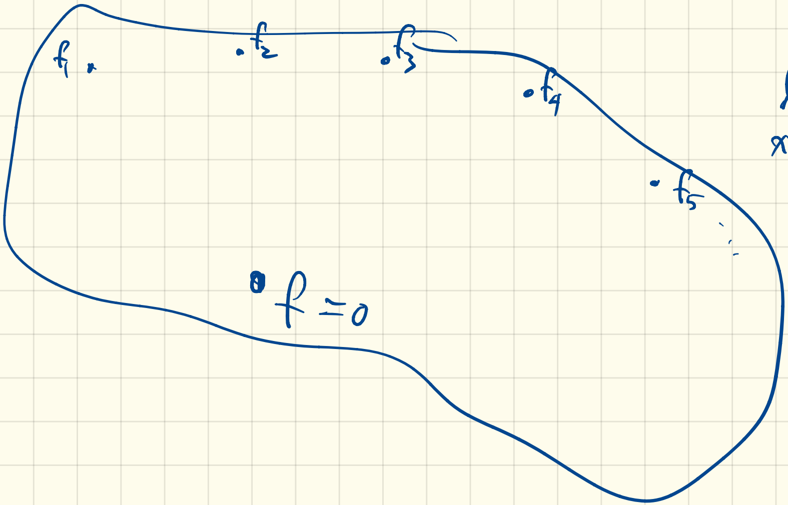
$$\| f_n - f \| = \sup_{x \in [0,1]} | f_n(x) - f(x) | = 1$$

For $0 < x < 1$, $|x^n - 0| = x^n$

$$\sup_x x^n = 1.$$



$$\lim_{x \to 1^-} x^n = 1 \quad \text{for each } n \in \mathbb{N}.$$



The terms $f_1, f_2, f_3, f_4, \ldots$ do not approach $f = 0$. They have distance 1 away from f.

Eg. The Taylor series for e^x is $T(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$

The partial sums are the Taylor polynomials $T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$.

For all x, $T(x)$ converges (absolutely) to e^x.

Actually, $T(x) \to e^x$ pointwise on $\mathbb{R}$, not uniformly.

This means by definition that the sequence of partial sums $T_n(x)$ converges pointwise to e^x on $\mathbb{R}$.

For all x, $\lim_{n\to\infty} T_n(x) = e^x$. This convergence is not uniform on $\mathbb{R}$.

(How many terms do we need for $T_n(x)$ to agree with e^x within ε ?

This depends on how big x is. If x is close to zero, only a few terms are needed. For $|x|$ large, many more terms in the series are needed.)

More concisely, the convergence $T_n(x) \to e^x$ is uniform on $[a,b]$ i.e. on compact subsets of $\mathbb{R}$ but not on $\mathbb{R}$.

Theorem (Weierstrass M-test) Let $f_n : A \to \mathbb{R}$ be a sequence of functions satifying $|f_n(x)| \le M_n$ for all $x \in A$, $n \geq 1$ where $\sum M_n < \infty$. Then $\sum f_n$ converges uniformly and absolutely on A.

Proof Recall: convergence of $\sum f_n$ refers to convergence of the sequence of partial sums $S_n(x) = \sum_{k=1}^{n} f_k(x)$. We first verify that for each $x \in A$, the series converges. For $m > n$

$$|S_m(x) - S_n(x)| = |f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x)| \le |f_{n+1}(x)| + |f_{n+2}(x)| + \cdots + |f_m(x)|$$

$$\le M_{n+1} + M_{n+2} + \cdots + M_m = |s_m - s_n| \qquad \text{where } s_n = M_1 + M_2 + \cdots + M_n.$$

Given $\varepsilon > 0$, there exists N such that $|s_m - s_n| < \varepsilon$ whenever $m, n > N$.

In this case, for all $x \in A$, $|S_m(x) - S_n(x)| < \varepsilon$ for all $m, n > N$.

For each fixed $x \in A$, $(S_m(x))_m$ is a sequence of numbers depending on x, satifying the Cauchy criterion. So this sequence converges to some value $S(x)$ depending on $x \in A$. That is, $S_m(x) \to S(x)$ converges (pointwise) for each $x \in A$.

Now we just need to prove that the convergence is uniform.

Let $\varepsilon > 0$. Since $s_n \to s = \sum_n M_n$ as $n \to \infty$, there exists N such that $|s_n - s| < \varepsilon$ whenever $n > N$. Then

$$\left|S(x) - S_n(x)\right| = \left|\lim_{m\to\infty}\left(S_m(x) - S_n(x)\right)\right| = \lim_{m\to\infty}\left|S_m(x) - S_n(x)\right| \le \lim_{m\to\infty}|s_m - s_n|$$

$$= |s - s_n| < \varepsilon \quad \text{whenever } n > N \text{ and } x \in A.$$

This says $S_n(x) \to S(x)$ uniformly and absolutely for $x \in A$. $\square$

Eg. $T(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges absolutely pointwise to e^x ~~on~~ $\mathbb{R}$. The convergence is not uniform on $\mathbb{R}$ but it is uniform on closed intervals $[a,b]$ and more generally on compact subsets of $\mathbb{R}$.

The convergence cannot be uniform ~~on~~ $\mathbb{R}$. If it were, then there would exist N such that $|T_n(x) - e^x| < 1$ for all $x \in \mathbb{R}$, $n > N$. Here $T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$ is a polynomial of degree n. This cannot hold since e^x grows faster than any polynomial as $x \to \infty$. For example, it would say

$$\lim_{x\to\infty} \frac{T_n(x) - e^x}{e^x} = 0$$ by the Squeeze Theorem. This contradicts

$$\lim_{x\to\infty} \frac{T_n(x) - e^x}{e^x} = \lim_{x\to\infty}\left(\frac{T_n(x)}{e^x} - 1\right) = 0 - 1 = -1.$$

On $[a,b]$, however, the convergence $T_n(x) \longrightarrow e^x$ is uniform.

$$T_n(x) = \sum_{k=0}^{k} \underbrace{\frac{x^k}{k!}}_{f_k(x)} \quad \text{where} \quad |f_k(x)| = \left|\frac{x^k}{k!}\right| \le M_k \quad \text{where} \quad M_k = \frac{r^k}{k}, \; r = \max\{|a|, |b|\}$$

$$\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} \frac{r^k}{k!} = e^r < \infty.$$ So $T(x) \to e^x$ uniformly and absolutely on $[a,b]$.

<u>Another example</u>: $\sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$ converges uniformly and absolutely on $\mathbb{R}$.

Here $\left|\frac{1}{n^2+x^2}\right| = \frac{1}{n^2+x^2} \le \frac{1}{n^2}$ for all x, $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ by the "p-series test".

<u>Remark</u> $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Compare: $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ converges to a known value (found earlier in the course).