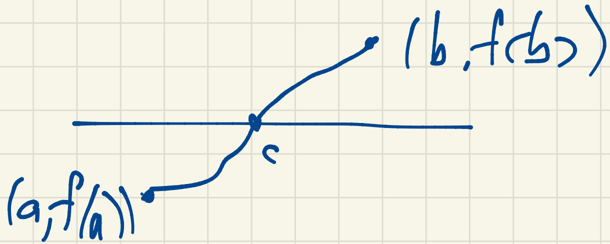
Analysis I (Math 3205)
Fall 2020

Book I

Intermediate Value Theorem

If $f : [a,b] \to \mathbb{R}$ is continuous with $f(a) < 0 < f(b)$, then there exists $c \in (a,b)$ such that $f(c) = 0$.

How does anyone prove this?
what is *your* experience with reading / writing proofs?



The theorem does not hold over $\mathbb{Q}$.

eg. $f(x) = x^2 - 2$, $f : \{\text{rationals between } 0 \text{ and } 2\} \to \mathbb{Q}$ is continuous, $f(0) < 0 < f(2)$ but there is no solution of $f(c) = 0$ in $\mathbb{Q}$.

$\mathbb{Q}$ is not complete; $\mathbb{R}$ is complete.

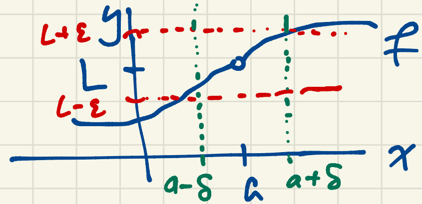
The complete statement of the Intermediate Value Theorem: For all $f : [a,b] \to \mathbb{R}$, if f is continuous and $f(a) < 0 < f(b)$, then there exists $c \in (a,b)$ such that $f(c) = 0$.

"for all", "for every", "for each" : universal quantifiers

"there is", "there exists" : existential quantifiers.

For all x there exists y such that $x < y$. (True in $\mathbb{R}$)

There exists y such that for all x, $x < y$. (False in $\mathbb{R}$)



Note: It doesn't matter here what $f(a)$ is or even whether or not it's defined

Definition of a Limit

We say $\lim_{x \to a} f(x) = L$ if the following condition holds:

For all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{|f(x) - L| < \varepsilon}_{f(x) \text{ is within } \varepsilon \text{ of } L} \quad \text{whenever} \quad \underbrace{0 < |x-a| < \delta}_{x \text{ is within } \delta \text{ of } a}.$$

i.e. for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all x, if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

Let's prove that $\lim_{x\to 2}(5x+1) = 11$.

Rough version: $f(x) = 5x+1$. If we need $f(x)$ to be within ε of 11, how close does x have to be to 2 ?

$$|f(x) - 11| < \varepsilon \iff 11-\varepsilon < f(x) < 11+\varepsilon$$
$$\iff 11-\varepsilon < 5x+1 < 11+\varepsilon$$
$$\iff 10-\varepsilon < 5x < 10+\varepsilon$$
$$\iff 2-\frac{\varepsilon}{5} < x < 2+\frac{\varepsilon}{5}$$
$$\iff |x-2| < \frac{\varepsilon}{5}$$

Proof (Actually): Let $\varepsilon > 0$. Then whenever $0 < |x-2| < \frac{\varepsilon}{5}$ we have $2-\frac{\varepsilon}{5} < x < 2+\frac{\varepsilon}{5}$ so

$11-\varepsilon < 5x+1 < 11+\varepsilon$ i.e. $|f(x)-11| < \varepsilon$.

Another proof: Suppose $\lim_{x\to 7} f(x) = 4$ and $\lim_{x\to 7} g(x) = 5$. Prove that $\lim_{x\to 7}(f(x)+g(x)) = 9$.

Rough version: Given $\varepsilon > 0$ we must find $\delta > 0$ such that $|f(x)+g(x)-9| < \varepsilon$ whenever $0 < |x-7| < \delta$. Since $\lim_{x\to 7} f(x) = 4$, we can find $\delta > 0$ such that $|f(x)-4| < \varepsilon$ whenever $0 < |x-7| < \delta$. Also since $\lim_{x\to 7} g(x) = 5$, we can find δ' such that

$|g(x) - 7| < \varepsilon$ whenever $0 < |x-7| < \delta'$.

$4-\varepsilon < f(x) < 4+\varepsilon$ whenever $7-\delta < x < 7+\delta$

$4-\varepsilon < f(x) < 4+\varepsilon$ whenever $7-\delta < x < 7+\delta$ i.e. $|x-7|<\delta$

$5-\varepsilon < g(x) < 5+\varepsilon$ whenever $7-\delta' < x < 7+\delta'$ i.e. $|x-7|<\delta'$

$\underbrace{9-2\varepsilon < f(x)+g(x) < 9+2\varepsilon}$ whenever $0<|x-7| < \min\{\delta, \delta'\}$

$|f(x)+g(x)-9| < 2\varepsilon$ whenever $0<|x-7| < \min\{\delta, \delta'\}$.

Actual (final) proof:

Let $\varepsilon>0$. There exists δ such that $|f(x)-4| < \frac{\varepsilon}{2}$ whenever $0<|x-7|<\delta$. Also there exists $\delta'>0$ such that $|g(x)-5|<\frac{\varepsilon}{2}$ whenever $0<|x-7|<\delta'$. Then

$$|f(x)+g(x)-9| \leq |f(x)-4| + |g(x)-5| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

whenever $0<|x-7|<\min\{\delta, \delta'\}$. $\square$

Note: The triangle inequality says $|a+b| \leq |a|+|b|$ for all a, b.

$|f(x)+g(x)-9| = |f(x)-4+g(x)-5|$

$|f(x)-4+g(x)-5| \leq |f(x)-4| + |g(x)-5|$

$|f(x)+g(x)-9| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}$

$|f(x)+g(x)-9| < \varepsilon$

$7-2=5 \Longleftrightarrow 7-2=\boxed{5}$

$\Longleftrightarrow 7=\boxed{5}+2$

$\infty+1=\infty \Longleftrightarrow \infty-\infty=1$

$\infty+2=\infty \Longleftrightarrow \infty-\infty=2$

$\Big\} \Rightarrow 1=2.$

$\lim_{x\to\infty} f(x)=\infty$ <u>means</u> for all M there exist N such that $f(x)>M$ whenever $x>N$.

$[N,\infty)=\{a\in\mathbb{R} : a\geq N\}.$

$(N,\infty)=\{a\in\mathbb{R} : a>N\}.$

$0.9999999\ldots = 1$

$\frac{1}{3}=0.333333\ldots$