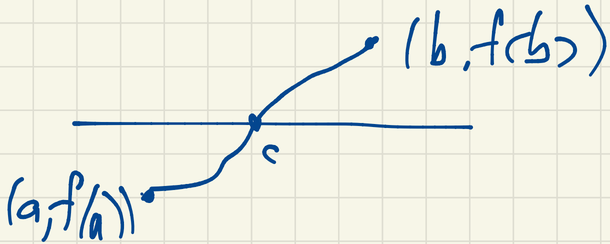
Analysis I (Math 3205)
Fall 2020

Book I

Intermediate Value Theorem

If $f: [a,b] \to \mathbb{R}$ is continuous with $f(a) < 0 < f(b)$, then there exists $c \in (a,b)$ such that $f(c) = 0$.

How does anyone prove this?
What is *your* experience with reading/writing proofs?



The theorem does not hold over $\mathbb{Q}$.

eg. $f(x) = x^2 - 2$, f: {rationals between 0 and 2} $\longrightarrow \mathbb{Q}$ is continuous, $f(0) < 0 < f(2)$ but there is no solution of $f(c) = 0$ in $\mathbb{Q}$.

$\mathbb{Q}$ is not complete; $\mathbb{R}$ is complete.

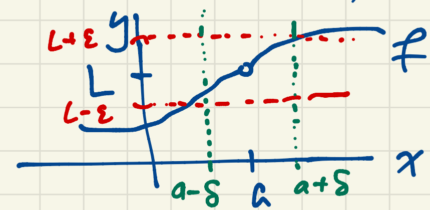
The complete statement of the Intermediate Value Theorem: For all $f: [a,b] \to \mathbb{R}$, if f is continuous and $f(a) < 0 < f(b)$, then there exists $c \in (a,b)$ such that $f(c) = 0$.

"for all", "for every", "for each" : Universal quantifiers

"there is", "there exists": existential quantifiers.

For all x **there exists y** such that $x < y$. (True in $\mathbb{R}$)

There exists y such that **for all x,** $x < y$. (False in $\mathbb{R}$)



Definition of a Limit

We say $\lim_{x\to a} f(x) = L$ if the following condition holds:

For all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{|f(x) - L| < \varepsilon}_{f(x)\text{ is within }\varepsilon\text{ of }L} \quad \text{whenever} \quad \underbrace{0 < |x-a| < \delta}_{x\text{ is within }\delta\text{ of }a}.$$

i.e. **for all $\varepsilon>0$**, **there exists $\delta>0$** such that **for all x,** if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$.

Note: It doesn't matter here what $f(a)$ is or even whether or not it's defined.

Let's prove that $\lim_{x \to 2} (5x+1) = 11$.

Rough version: $f(x) = 5x+1$. If we need $f(x)$ to be within ε of 11, how close does x have to be to 2?

$$
\begin{aligned}
|f(x) - 11| < \varepsilon &\iff 11-\varepsilon < f(x) < 11+\varepsilon \\
&\iff 11-\varepsilon < 5x+1 < 11+\varepsilon \\
&\iff 10-\varepsilon < 5x < 10+\varepsilon \\
&\iff 2-\frac{\varepsilon}{5} < x < 2+\frac{\varepsilon}{5} \\
&\iff |x-2| < \frac{\varepsilon}{5}
\end{aligned}
$$

Proof (~~Actually~~): Let $\varepsilon > 0$. Then whenever $0 < |x-2| < \frac{\varepsilon}{5}$ we have $2-\frac{\varepsilon}{5} < x < 2+\frac{\varepsilon}{5}$ so $11-\varepsilon < 5x+1 < 11+\varepsilon$ i.e. $|f(x)-11| < \varepsilon$.