University of Wyoming

Math 3205—Fall 2020

Analysis I

Department of Mathematics

HW4

Due 12:15pm Thursday, December 10, 2020

This homework assignment is optional. If you do better on HW4 than your average on the other homework, this grade will be included in your homework grade. If you do better on the previous homework (or choose not to do HW4 at all) then your homework grade will be based solely on HW1,2,3.

Show your work. You may discuss homework with others, but *what you write must be your own work.*

1. (*20 points*) Consider the series $\sum_{n=1}^{\infty} \frac{x^2}{n^2+x^2}$.
   (a) Show that the given series converges pointwise for all $x \in \mathbb{R}$.
   (b) Show that the given series converges absolutely and uniformly on every interval $[a,b]$.
   (c) Is the convergence uniform on $\mathbb{R}$? Explain.

2. (*15 points*) Let $A, B \subset \mathbb{R}$ be compact sets. Suppose there are sequences of points $(a_n)_n$ in A, and $(b_n)_n$ in B, such that $|a_n - b_n| < \frac{1}{n}$ for every positive integer n. Show that $A \cap B$ is nonempty.

3. (*25 points*) For each positive integer n, define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \frac{nx}{(1+nx^2)^2}$.
   (a) Show that there is a pointwise limit function $f(x) = \lim_{n\to\infty} f_n(x)$ defined for all x; and evaluate $f(x)$ in simplified form.
   (b) Does $f_n \to f$ uniformly on $\mathbb{R}$? Explain.
   (c) Does $f_n \to f$ uniformly on closed intervals $[a,b] \subset \mathbb{R}$? Better yet, on which closed intervals $[a,b]$ is the convergence uniform?
   (d) Evaluate $\int_0^1 f_n(x)\,dx$ and $\int_0^1 f(x)\,dx$.
   (e) Does $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx$? Explain.