UNIVERSITY OF WYOMING

Math 3205—Fall 2020

Analysis I

Department of Mathematics

HW3

Due 5:00 pm Friday, November 20, 2020

This assignment is graded out of 60, with 10 bonus points available. (The total grade will be capped at 60 points.)

1. (*15 points*) Let $A = \left\{\frac{mn}{m+n} : m, n \text{ are positive integers}\right\}$.
   (a) Find a sequence of distinct points a_n in A converging to 10.
   (b) Determine the derived set A' (the set of limit points of A).
   (c) Is the set A closed? Justify your answer.

2. (*10 points*) Let $f, g : \mathbb{R} \to \mathbb{R}$ and suppose $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$. Does it follow necessarily that $\lim_{x\to 0} f(g(x)) = 0$? Prove this (from the definitions!) or provide a counterexample.

3. (*10 points*) Let $A \subseteq \mathbb{R}$ be a nonempty open set. Show that there exists a sequence of open intervals (a_n, b_n) (for $n = 1, 2, 3, \ldots$) such that $A = \bigcup_{n=1}^{\infty}(a_n, b_n)$.

   Remarks: In #3, you are free to use either of the two equivalent definitions we have given for open sets:
   (i) A set $A \subseteq \mathbb{R}$ is *open* if for every $x \in A$, there exists an open interval $(a, b) \subseteq A$ containing x.
   (ii) A set $A \subseteq \mathbb{R}$ is *open* if it is a union of some collection of open intervals.

   Note however that (ii) gives you a possibly uncountable collection of open intervals, from which some work will be required to express A also as a union of a countable collection of open intervals.

4. (*15 points*) Let (a_n) and (b_n) be sequences of real numbers such that $|a_{n+1} - a_n| \leqslant b_n$ for all n. Suppose that $\sum_n b_n$ converges.
   (a) Show that $|a_m - a_n| \leqslant \sum_{k=n}^{m-1} b_k$ whenever $m > n$.
   (b) Show that the sequence (a_n) is Cauchy.
   (c) Show that the sequence (a_n) converges.

5. (*20 points*) Indicate whether each of the following sets is closed, open, both, or neither (in the standard topology of $\mathbb{R}$).

   (a) $\{x \in \mathbb{R} : x \sin x > 5\}$
   (b) The set of all irrational real numbers.
   (c) The set of all rational numbers having denominator at most 100.
   (d) The set of all real numbers having a decimal expansion containing the digit 7.