UNIVERSITY OF WYOMING

Math 3205—Fall 2020

Analysis I

Department of Mathematics

Solutions to HW1

In #1 there is more than one way to write answers, since intermediate steps can be shown in either ternary to decimal notation. Some will prefer to write the subscript '(3)' to indicate ternary expressions. I will instead say in words which notation I am using, and I will also use red font to indicate ternary expansions.

First Solution to #1(a) (using decimal notation to show the actual work.)
We are given the number

$$a = 1{\cdot}3^3 + 0{\cdot}3^2 + 1{\cdot}3^1 + 2{\cdot}3^0 + 1{\cdot}3^{-1} + 0{\cdot}3^{-2} + 2{\cdot}3^{-2} + 1{\cdot}3^{-4} + 0{\cdot}3^{-5} + 2{\cdot}3^{-6} + \cdots.$$

Multiply by $3^3 = 27$ to get

$$27a = 1{\cdot}3^6 + 0{\cdot}3^5 + 1{\cdot}3^4 + 2{\cdot}3^3 + 1{\cdot}3^2 + 0{\cdot}3^1 + 2{\cdot}3^0 + 1{\cdot}3^{-1} + 0{\cdot}3^{-2} + 2{\cdot}3^{-3} + \cdots.$$

Subtract to get

$$26a = (1{\cdot}3^6 + 0{\cdot}3^5 + 1{\cdot}3^4 + 2{\cdot}3^3 + 1{\cdot}3^2 + 0{\cdot}3^1 + 2{\cdot}3^0) - (1{\cdot}3^3 + 0{\cdot}3^2 + 1{\cdot}3^1 + 2{\cdot}3^0) = 843$$

so $a = \frac{843}{26}$.

Second Solution to #1(a) (using ternary notation to show the actual work.)

$$\begin{array}{rl} 1000a = & 1012102.102102\ldots \\ a = & 1012.102102\ldots \\ \hline 222a = & 1011020.000000\ldots \end{array} \qquad \text{so } a = \frac{1011020}{222} = \frac{843}{26}.$$

As stated in the Decimals video, a real number has two distinct decimal expansions iff it is a rational number whose reduced form has denominator consisting of a power of 2 times a power of 5. Here 2 and 5 are the prime divisors of the base 10. Analogously, those real numbers having two distinct ternary expansions are the rational numbers whose reduced form has denominator consisting of a power of 3. You were not asked for proofs of these facts.

Solution to #1(b)

The rational number $\frac{1}{3} = 0.33333333\ldots$ has a unique decimal expansion. Its two ternary expansions are

$$0.1000000\ldots = 0{\cdot}3^0 + 1{\cdot}3^{-1} + 0{\cdot}3^{-2} + 0{\cdot}3^{-3} + 0{\cdot}3^{-4} + \cdots$$

and

$$0.0222222\ldots = 0{\cdot}3^0 + 0{\cdot}3^{-1} + 2{\cdot}3^{-2} + 2{\cdot}3^{-3} + 2{\cdot}3^{-4} + \cdots.$$

Solution to #1(c)

The rational number $\frac{1}{2} = 0.500000000\ldots = 0.499999999\ldots$ has two decimal expansions. Its unique ternary expansion is

$$0.111111111\ldots = 0{\cdot}3^0 + 1{\cdot}3^{-1} + 1{\cdot}3^{-2} + 1{\cdot}3^{-3} + 1{\cdot}3^{-4} + \cdots.$$

Solution to #1(d)

The rational number $1 = 1.000000000\ldots = 0.999999999\ldots$ has two decimal expansions. Its two ternary expansions are

$$1.0000000\ldots = 1{\cdot}3^0 + 0{\cdot}3^{-1} + 0{\cdot}3^{-2} + 0{\cdot}3^{-3} + 0{\cdot}3^{-4} + \cdots$$

and

$$0.2222222\ldots = 0{\cdot}3^0 + 2{\cdot}3^{-1} + 2{\cdot}3^{-2} + 2{\cdot}3^{-3} + 2{\cdot}3^{-4} + \cdots.$$

Solution to #1(e)

The rational number $\frac{1}{6} = 0.166666666\ldots$ has a unique decimal expansion and a unique ternary expansion

$$0.011111111\ldots = 0{\cdot}3^0 + 0{\cdot}3^{-1} + 1{\cdot}3^{-2} + 1{\cdot}3^{-3} + 1{\cdot}3^{-4} + 1{\cdot}3^{-5} + \cdots.$$

Solution to #2

Let $\varepsilon > 0$ be given. We may choose $\delta = \min\{1, \frac{\varepsilon}{7}\}$. Whenever $0 < |x-3| < \delta$, we have $x \in (2,4)$ and $|x+3| < 7$, so

$$|x^2 - 9| = |x+3||x-3| < 7{\cdot}\tfrac{\varepsilon}{7} = \varepsilon.$$

Thus $\lim\limits_{x\to 3} x^2 = 9$.

Solution to #3

(i) Using l'Hôpital's Rule,

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{\frac{\sin h}{h} - 1}{h} = \lim_{h\to 0} \frac{\sin h - h}{h^2} = \lim_{h\to 0} \frac{\cos h - 1}{2h} = \lim_{h\to 0} \frac{-\sin h}{2} = 0.$$

When $x \neq 0$, we have $f'(x) = \frac{x\cos x - \sin x}{x^2}$ and $f''(x) = \frac{(2-x^2)\sin x - 2x\cos x}{x^3}$. Also

$$\begin{aligned} f''(0) &= \lim_{h\to 0} \frac{f'(h) - f'(0)}{h} = \lim_{h\to 0} \frac{\frac{h\cos h - \sin h}{h^2} - 0}{h} = \lim_{h\to 0} \frac{h\cos h - \sin h}{h^3} = \lim_{h\to 0} \frac{-h\sin h}{3h^2} \\ &= \lim_{h\to 0} -\frac{\sin h}{3h} = \lim_{h\to 0} -\frac{\cos h}{3} = -\frac{1}{3} \end{aligned}$$

and so $f^{(n)}(0) = 1, 0, -\frac{1}{3}$ for $n = 0, 1, 2$. (*Remark:* From this you can see that $f \in C^2(\mathbb{R})$, although this information was not required.)

(ii) The second order Taylor polynomial for $f(x)$ centered at 0 is $T_2(x) = 1 + 0x + \frac{1}{2}\left(-\frac{1}{3}\right)x^2 = 1 - \frac{1}{6}x^2$.

Check: Using the Taylor series $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$, the Taylor series for $f(x)$ is seen to be $1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \cdots$.