Solutions to HW4

1. Let $f_n(x) = \frac{x^2}{n^2+x^2}$ and $f(x) = \sum\limits_{n=1}^{\infty} f_n(x)$. Note that all values of these functions are non-negative.

   (a) For each $x \in \mathbb{R}$, the series for $f(x)$ converges (absolutely) since $f(x) \leqslant x^2 \sum_n \frac{1}{n^2} = \frac{\pi^2 x^2}{6}$. (Here we have used the value $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$; but all that we actually need is the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.) In particular, $f : \mathbb{R} \to \mathbb{R}$ is a well-defined function.

   (b) Let $r = \max\{|a|, |b|\}$, so that $[a, b] \subseteq [-r, r]$. Clearly $|f_n(x)| \leqslant \frac{r^2}{n^2}$ for all $n \geqslant 1$ and all $x \in [a, b]$. Since $\sum\limits_{n=1}^{\infty} \frac{r^2}{n^2} = \frac{\pi^2 r^2}{6}$ converges, by the Weierstrass M-test the series $f(x) = \sum_n f_n(x)$ converges uniformly and absolutely on $[a, b]$.

   (c) The series $\sum_n f_n(x)$ does *not* converge uniformly on $\mathbb{R}$. We show this by contradiction using $\varepsilon = \frac{1}{2}$. We denote $S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$. If $S_n \to f$ uniformly on $\mathbb{R}$, then there exists N such that $|S_n(x) - f(x)| < \frac{1}{2}$ for all $n > N$ and all $x \in \mathbb{R}$. In particular, we have $\sum_{n>N} f_n(x) < \frac{1}{2}$ for all $x \in \mathbb{R}$. Now choose an integer $n > N$ and fix a value of $x > n$ to get

   $$f_n(x) = \frac{x^2}{n^2 + x^2} = \frac{1}{\frac{n^2}{x^2} + 1} \geqslant \frac{1}{1+1} = \frac{1}{2},$$

   a contradiction.

2. Since A is compact, the sequence $(a_n)_n$ has a convergent subsequence $(a_{n_k})_k \to a \in A$. Here $(n_k)_k$ is an increasing sequence of positive integers, and in particular $(n_k)_k \to \infty$. Now

   $$|b_{n_k} - a| \leqslant |b_{n_k} - a_{n_k}| + |a_{n_k} - a| \leqslant \tfrac{1}{n_k} + |a_{n_k} - a| \to 0$$

   as $k \to \infty$. Thus $(b_{n_k})_k \to a$. Since B is also compact, this proves that $a \in B$. Since $a \in A \cap B$, we have $A \cap B \neq \varnothing$.

3. (a) We have $f(x) = \lim_{n\to\infty} \frac{nx}{(1+nx^2)^2} = \frac{0}{1} = 0$ by the limit laws.

   (b) The convergence $f_n \to f$ cannot be uniform on $\mathbb{R}$, as we show by contradiction. If the convergence were uniform, then for all $\varepsilon > 0$, there would exist N such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > N$ and all $x \in \mathbb{R}$. In particular for $\varepsilon = 0.1$, this means that there exists N such that $|f_n(x)| < 0.1$ for all $n > N$ and all $x \in \mathbb{R}$. In this case choose a positive integer $n > N$ and take $x = \frac{1}{n}$ to obtain $f_n(x) = \frac{1}{(1+\frac{1}{n})^2} \geqslant \frac{1}{4}$. This contradicts $|f_n(x)| < 0.1$.

(c) Exactly the same argument as in (b) shows that the convergence $f_n \to f$ cannot be uniform on any closed intervals containing 0 (i.e. intervals $[a, b]$ with $a < b$ and $a \leqslant 0 \leqslant b$) since such intervals contain infinitely many points of the form $\pm\frac{1}{n}$. However, convergence is uniform on closed intervals *not* containing 0. We consider a closed interval $[a, b]$ or $[a, \infty)$ with $a > 0$. (The same argument works on closed subsets of $(-\infty, 0)$ since each f_n is an odd function.) Given $\varepsilon > 0$ and $a > 0$, take $N = \frac{1}{a^3\varepsilon}$. Whenever $n > N$ and $x \geqslant a$, we have

$$|f_n(x)| = \frac{nx}{(1 + nx^2)^2} < \frac{nx}{(nx^2)^2} = \frac{1}{nx^3} \leqslant \frac{1}{na^3} < \varepsilon$$

which shows that the convergence $f_n \to 0$ is uniform on $[a, \infty)$ and on $[a, b]$.

(d) Substituting $u = 1+nx^2$ gives $\int_0^1 f_n(x)\,dx = \int_0^1 \frac{nx\,dx}{(1+nx^2)^2} = \frac{1}{2}\int_1^{1+n} u^{-2}du = \left[-\frac{1}{2u}\right]_1^{1+n} = \frac{1}{2}\left(1 - \frac{1}{n+1}\right) = \frac{n}{2(n+1)}$. Also $\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$.

(e) No; we have $\int_0^1 f_n(x)\,dx = \frac{n}{2(n+1)} \to \frac{1}{2}$ as $n \to \infty$. This differs from $\int_0^1 f(x)\,dx = 0$. This independently confirms that the convergence $f_n \to f$ is not uniform.