UNIVERSITY OF WYOMING

Math 3205—Fall 2020

Analysis I

Department of Mathematics

Solutions to HW3

1. It is useful to observe that the minimum element of A is $\frac{1}{2}$. (If $m, n \in \mathbb{N}$ then $\frac{m+n}{mn} = \frac{1}{m} + \frac{1}{n} \leqslant 1 + 1 = 2$ so $\frac{mn}{m+n} \geqslant \frac{1}{2}$.)

   (a) The sequence $\left(\frac{10n}{10+n}\right)_n$ of distinct points in A converges to $10 \in A'$.

   (b) We show that $A' = \mathbb{N} = \{1, 2, 3, \ldots\}$. Since each $m \in \mathbb{N}$ is the limit of a sequence of distinct points $\left(\frac{mn}{m+n}\right)_n$ in A, we get $m \in A'$. This proves $\mathbb{N} \subseteq A'$.

   Conversely, suppose $x \in A'$. Then x is a limit of a sequence $(x_k)_k$ in A, where $x_k = \frac{m_k n_k}{m_k + n_k}$. Here $(m_k)_k$ and $(n_k)_k$ are sequences in $\mathbb{N}$. Since $(x_k)_k$ has infinitely many distinct terms, the sequences (m_k) and (n_k) cannot both be bounded; so without loss of generality, $(n_k)_k$ is unbounded. Thus $(n_k)_k$ has a strictly increasing subsequence. Without loss of generality, $(n_k)_k$ is increasing; otherwise we may replace it by an increasing subsequence. Since x is positive, we may consider

   $$\frac{1}{x} = \lim_{k\to\infty} \frac{1}{x_k} = \lim_{k\to\infty} \left(\frac{1}{m_k} + \frac{1}{n_k}\right) = \lim_{k\to\infty} \frac{1}{m_k}$$

   since $(n_k)_k \to \infty$. Thus $(m_k)_k$ is a convergent sequence of positive integers. This means that $(m_k)_k$ is constant for all sufficiently large k. In other words, there exist positive integers m, N such that $m_k = m$ for all $k > N$. This gives $x = m \in \mathbb{N}$ as claimed.

   (c) Each positive integer $k \in \mathbb{N}$ has the form $k = \frac{(2k)(2k)}{2k+2k} \in A$. Thus $A \supseteq \mathbb{N} = A'$, so A is closed.

2. This property fails. For example, consider

   $$f(x) = g(x) = \begin{cases} 0, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$$

   Then $\lim_{x\to 0} f(g(x)) = 1$ whereas $f(g(0)) = 0$. Based on the limit laws (from any calculus textbook), it's clear that a counterexample requires discontinuity; and in fact when you try discontinuous functions, it is easy to find lots of counterexamples.

3. Every open set $A \subseteq \mathbb{R}$ is a union of open intervals of the form (r, s) with rational endpoints r, s. Indeed, if $x \in A$ then there is an open interval $(a, b) \subseteq A$ containing x (by condition (i) as stated in the assignment); then we can find rational numbers $r \in (a, x)$ and $s \in (x, b)$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. This gives $x \in (r, s) \subseteq A$. Choosing

such an open interval (r,s) for each $x \in A$ allows us to express A as a union of open intervals with rational endpoints.

All that remains is to observe that there are only countably many intervals (r,s) with rational endpoints, since there are only countably many choices of a and b in $\mathbb{Q}$. In order to fully express this, however, I will want to carefully distinguish the ordered pair (r,s) from the open interval (r,s). (Unfortunately we use the same notation for both concepts; so I will say in words to which I am referring in each case.) There are countably many ordered pairs (r,s) since there are countably many choices for r and countably many choices for s. (As in the video on Cardinality, we can enumerate $\mathbb{Q} = \{a_1, a_2, a_3, \ldots\}$ and then consider an infinite 'table' with the ordered pair (a_m, a_n) in rom m and column n. There is a 'snake-like' path through all the entries of the table, showing us that there is a sequence containing all ordered pairs $(a_m, a_n) \in \mathbb{Q}^2$.) Now if we omit all the ordered pairs (a_m, a_n) for which $a_m \geqslant a_n$, we are left with a countable set of ordered pairs (a_m, a_n) having $a_m < a_n$. These ordered pairs define the set of all open intervals having rational endpoints; and so there are countably many such intervals. So A contains only a countable number of open intervals with rational endpoints.

4. (a) Whenever $m > n$,

$$|a_m - a_n| = \left| \sum_{k=n}^{m-1} (a_{k+1} - a_k) \right| \leqslant \sum_{k=n}^{m-1} \left| a_{k+1} - a_k \right| \leqslant \sum_{k=n}^{m-1} b_k.$$

(b) Let $\varepsilon > 0$. Since $\sum_n b_n$ is a convergent series with positive terms, its partial sums $s_n = \sum_{k \leqslant n} b_k$ form a bounded weakly increasing sequence with limit L. So we may choose $N > 0$ such that $L - \varepsilon < s_N \leqslant L$. For all numbers $m > n > N$, we have

$$|a_m - a_n| \leqslant \sum_{k=n}^{m-1} b_k = s_m - s_n < \varepsilon.$$

This inequality holds more generally whenever $m, n > N$, so the sequence $(a_n)_n$ is Cauchy.

(c) Since $\mathbb{R}$ is complete, $(a_n)_n$ converges.

5. It is easy to see in each case (a)–(d) that the indicated set is neither empty nor $\mathbb{R}$, so it cannot be both open and closed.

(a) This set is open since it is $f^{-1}((5,\infty))$, the preimage of an open set under a continuous function $f(x) = x \sin x$. It is not closed for the reasons indicated above ($0 \notin f^{-1}((5,\infty))$ and $\frac{5\pi}{2} \in f^{-1}((5,\infty))$).

(b) Since $\mathbb{Q}$ is neither open nor closed, its complement $\mathbb{R}-\mathbb{Q}$ is also neither open nor closed. Again this is because both $\mathbb{Q}$ and $\mathbb{R}-\mathbb{Q}$ are dense in $\mathbb{R}$. (Every nonempty open interval contains both a rational and an irrational, so neither $\mathbb{Q}$ nor $\mathbb{R}-\mathbb{Q}$ contains a nonempty open set.)

(c) Let A be the set of all rational numbers having denominator at most 100. Then A is closed; and as indicated above, it cannot be open. Note that $A' = \varnothing$ since every point of A is isolated. (There are only finitely many elements of A in any bounded interval; so given $a \in A$, there exists $\delta > 0$ such that $(a-\delta, a+\delta)$ contains no points of $A - \{a\}$.) Alternatively, the complement $U = \mathbb{R} - A$ is open. This is because given any $u \in U$, there there exists $\delta > 0$ such that $(a-\delta, a+\delta)$ contains no points of A, i.e. $(a-\delta, a+\delta) \subset U$.

(d) Let S be the set of all real numbers having a decimal expansion containing the digit 7. Then $0 \notin S$ (since 0 has a unique decimal expansion $0.00000\ldots$, without the digit 7). And S contains a sequence $(s_n)_n \to 0$ where $s_n = 7 \times 10^{-n}$. Since S does not contain $0 \in S'$, S is not closed.

Also S is not open since its complement $T = \mathbb{R} - S$ contains a sequence $(b_n)_n \to 7$ where $b_n = 7 - 10^{-n}$. Each b_n has two decimal expansions, neither of which have 7 as a digit; for example, $b_3 = 6.9990000\ldots = 6.9989999\ldots.$ Since T does not contain $7 \in S$, T is not closed, so S is not open.