University of Wyoming — Math 3205—Fall 2020 — **Analysis I** — Department of Mathematics

Solutions to HW2

1. (a) The function f is continuous at $u \in \mathbb{R}$ iff u is irrational. To show this, it suffices to prove that $\lim_{x\to u} f(x) = 0$ for every $u \in \mathbb{R}$. So let $\varepsilon > 0$, there are only finitely many rationals in $(0, \varepsilon)$ having denominator $\geqslant \varepsilon$; and so we may choose $\delta > 0$ such that none of these rationals satisfy $0 < |x - u| < \delta$. So $0 \leqslant f(x) < \varepsilon$ whenever $0 < |x - u| < \delta$. This proves the result.

   (b) On every interval, f has minimum value 0 so the lower Riemann sum is 0. Here are the maximum values of f on the 4 required subintervals:

k	1	2	3	4
interval	$[0, \frac14]$	$[\frac14, \frac12]$	$[\frac12, \frac34]$	$[\frac34, 1]$
maximum	$f(0) = 1$	$f(\frac12) = \frac12$	$f(\frac12) = \frac12$	$f(1) = 1$

   so the upper Riemann sum is $\frac14\big(1 + \frac12 + \frac12 + 1\big) = 0.75$.

   (c) On every interval, f has minimum value 0 so the lower Riemann sum is 0. Here are the maximum values of f on the 20 required subintervals:

k	1	2	3	4	5	6	7
interval	$[0, .05]$	$[.05, .1]$	$[.1, .15]$	$[.15, .2]$	$[.2, .25]$	$[.25, .3]$	$[.3, .35]$
maximum	$f(0)$=1	$f(\frac{1}{10})$=$\frac{1}{10}$	$f(\frac17)$=$\frac17$	$f(\frac15)$=$\frac15$	$f(\frac14)$=$\frac14$	$f(\frac14)$=$\frac14$	$f(\frac13)$=$\frac13$

k	8	9	10	11	12	13	14
interval	$[.35, .4]$	$[.4, .45]$	$[.45, .5]$	$[.5, .55]$	$[.55, .6]$	$[.6, .65]$	$[.65, .7]$
maximum	$f(\frac25)$=$\frac15$	$f(\frac25)$=$\frac15$	$f(\frac12)$=$\frac12$	$f(\frac12)$=$\frac12$	$f(\frac35)$=$\frac15$	$f(\frac35)$=$\frac15$	$f(\frac23)$=$\frac13$

k	15	16	17	18	19	20
interval	$[.7, .75]$	$[.75, .8]$	$[.8, .85]$	$[85, .9]$	$[.9, .95]$	$[.95, 1]$
maximum	$f(\frac34)$=$\frac14$	$f(\frac34)$=$\frac14$	$f(\frac45)$=$\frac15$	$f(\frac67)$=$\frac17$	$f(\frac{9}{10})$=$\frac{1}{10}$	$f(1)$=1

   so the upper Riemann sum is

$$\tfrac{1}{20}\big(1+\tfrac{1}{10}+\tfrac17+\tfrac15+\tfrac14+\tfrac14+\tfrac13+\tfrac15+\tfrac15+\tfrac12+\tfrac12+\tfrac15+\tfrac15+\tfrac13+\tfrac14+\tfrac14+\tfrac15+\tfrac17+\tfrac{1}{10}+1\big) = \tfrac{667}{2100} \approx 0.31762.$$

   (d) In fact $\int_0^1 f(x)\,dx = 0$. With the Lebesgue integral, this fact holds because $f = 0$ almost everywhere, i.e. except on the rationals (a set of measure zero). This follows because $\mathbb{Q}$ is countable. But the Riemann integral is also zero. Unlike Dirichlet's function discussed in class (where $f(x) = 1$ for every rational number) which is not Riemann integrable, for the f in this problem, the infimum of the upper Riemann sums is zero so the integral is zero. You were *not* asked to prove this here, although it isn't hard; I have a included a proof below.

To show that $\int_0^1 f(x)\,dx = 0$, recall (as noted above) that all lower Riemann sums are zero. Given $\varepsilon > 0$, it suffices to show that we can find an upper Riemann sum less than ε. Let S be the set of rational numbers in $[0,1]$ having denominator less than $\frac{2}{\varepsilon}$ (in lowest terms). This is clearly a finite set; so take $N = |S| < \infty$. Take a positive integer $n > \frac{2N}{\varepsilon}$. Consider the partition of $[0,1]$ into n subintervals of equal width $\Delta x = \frac{1}{n}$. The corresponding upper Riemann sum has n positive terms, including

- at most N terms based on subintervals $[x_{i-1}, x_i]$ containing one or more points of S. Since each such subinterval has width $\frac{1}{n} < \frac{\varepsilon}{2N}$, the contribution of these terms to the upper Riemann sum is at most $N \cdot \frac{1}{n} < \frac{\varepsilon}{2}$. Also
- all remaining terms, based on subintervals $[x_{i-1}, x_i]$ not containing any point of S. These intervals have total width less than 1; and $f(x)$ is bounded above by $\frac{\varepsilon}{2}$ on these intervals (since any rational numbers in such intervals have denominator less than $\frac{2}{\varepsilon}$). So the total contribution to the upper Riemann sum coming from such terms is less than $\frac{\varepsilon}{2}$.

Since $\int_0^1 f(x)\,dx < \varepsilon$ for any positive ε, we must have $\int_0^1 f(x)\,dx = 0$.

2. (a) My calculator displays 0.739085133. Yours should give an answer very close to this.

   (b) $a_n = \begin{cases} 0, & \text{if } n = 0; \\ \cos(a_{n-1}), & \text{for } n = 1, 2, 3, \ldots. \end{cases}$

   (c) The function $g(x) = x - \cos x$ is strictly increasing since $g'(x) = 1 + \sin x \geqslant 0$ (and $g'(x) > 0$ except at isolated points $\pm\pi, \pm 3\pi, \pm 5\pi, \ldots$). Since g is strictly increasing, there is at most one solution of $g(x) = 0$. Since $g(0) = -1$ and $g(\frac{\pi}{2}) = 1$, g has a root $a \in (0,1)$ by the Intermediate Value Theorem. So this is the unique real root of $x = \cos x$.

   (d) The Mean Value Theorem for Derivatives asserts that for any differentiable function f and points $x \neq a$, there exists y between x and a such that $f(x) - f(a) = f'(y)(x - a)$. Applying this theorem to the case $f(x) = \cos x$ and $f'(x) = -\sin x$ gives the required result.

   (e) There is more than one choice of interval (and corresponding choice of c) which will satisfy the given conditions. But in order for (e) to be relevant to the induction argument in (f), we actually want an interval containing the terms of our sequence (a_n). For this purpose I will choose the interval $[0,1]$ on which the sine function is increasing. Thus $|\sin y| \leqslant \sin 1 \approx 0.84147$ for all $y \in [0,1]$. So we may choose $c = 0.85$ as a bound for the sine function on the interval $[0, 1]$. This interval contains $a \approx 0.739$; so assuming x is also in the interval $[0, 1]$, by (d) we have $|\cos x - a| \leqslant c|x - a|$. (Note here that $\cos a = a$.)

(f) We prove by induction that for all n, $a_n \in [0, 1]$ and $|a_n - a| \leqslant c^n a$. These conditions hold for $n = 0$ since $a_0 = 0 \in [0, 1]$ and $|a_0 - a| = a = c^0 a$. Now given any non-negative integer n for which the desired conditions hold, since $a_n \in [0, 1]$ we have $0 \leqslant a_n \leqslant \frac{\pi}{2}$ which yields $a_{n+1} = \cos a_n \in [0, 1]$. Moreover by (e),

$$|a_{n+1} - a| = |\cos(a_n) - a| \leqslant c|a_n - a| \leqslant c{\cdot}c^n a = c^{n+1} a.$$

So by induction, we have validated both of our claims.

(g) Since $|a_n - a| \leqslant c^n a$ for all n where $c \in (0, 1)$, we can make both sides arbitrarily small by choosing n sufficiently large. This gives an easy proof that $(a_n) \to a$. Alternatively, one may cite the Squeeze Theorem: The bounds in (f) are easily rewritten as $(1 - c^n)a \leqslant a_n \leqslant (1 + c^n)a$ for all n. Here the upper and lower bounds both converge to the same limit a as $n \to \infty$, so $(a_n) \to a$.

3. (a) This follows by contradiction. If $a_i \geqslant \frac{1}{n}$ for infinitely many values $i \in I$, then choosing these indices, we obtain a divergent series $a_{i_1} + a_{i_2} + a_{i_3} + \cdots$, contrary to the observations offered in the statement of the problem.

(b) As shown in (a), each of the sets $\{i \in I : a_i > \frac{1}{n}\}$ is finite. So $\bigcup_{n=1}^{\infty} \{i \in I : a_i > \frac{1}{n}\}$ is a countable union of finite sets, hence countable. (Indeed any countable union of countable sets is still countable, as indicated in the video on Cardinality.)

(c) Since a_i is positive for every $i \in I$, there is some natural number n for which $\frac{1}{n} < a_i$. (This follows from the Archimedean property as proved in the instructional video on the density of $\mathbb{Q}$ in $\mathbb{R}$.) So $I = \bigcup_{n=1}^{\infty} \{i \in I : a_i > \frac{1}{n}\}$, which we have seen to be countable.

4. (a) This series *converges* since it converges absolutely. Note here that $\left|\frac{\sin n}{n^2}\right| \leqslant \frac{1}{n^2}$ where $\sum_n \frac{1}{n^2}$ converges (this is a p-series with $p = 2 > 1$; or just use the Integral Test with $\int_1^{\infty} \frac{dx}{x^2} = 1 < \infty$).

(b) This series *converges*. Although the sequence $\left(\frac{n^{10}}{2^n}\right)$ is not decreasing, it is decreasing for $n > 14$; moreover, $\left(\frac{n^{10}}{2^n}\right) \to 0$. So we may disregard the first few terms and apply the Leibniz Test (also known as the Alternating Series Test).

(c) This series *diverges*. This is an example of a telescoping series; the partial sums exhibit cancellation leading to

$$
\begin{aligned}
s_n &= \sum_{k=1}^{n} \bigl(\sqrt{k+1} - \sqrt{k}\,\bigr) \\
&= \bigl(\cancel{\sqrt{2}} - 1\bigr) + \bigl(\cancel{\sqrt{3}} - \cancel{\sqrt{2}}\,\bigr) + \bigl(\cancel{\sqrt{4}} - \cancel{\sqrt{3}}\,\bigr) + \bigl(\cancel{\sqrt{5}} - \cancel{\sqrt{4}}\,\bigr) + \cdots + \bigl(\sqrt{n{+}1} - \cancel{\sqrt{n}}\,\bigr) \\
&= \sqrt{n{+}1} - 1 \to \infty
\end{aligned}
$$

as $n \to \infty$.

(d) This series *diverges*. As explained in the videos, $1{+}\frac{1}{2}{+}\frac{1}{3}{+}\cdots{+}\frac{1}{n} \leqslant 1 + \ln n$. All we require here is the much more obvious (and weaker) result that $1{+}\frac{1}{2}{+}\frac{1}{3}{+}\cdots{+}\frac{1}{n} \leqslant n$ which yields

$$\frac{1}{1{+}\frac{1}{2}{+}\frac{1}{3}{+}\cdots{+}\frac{1}{n}} \geqslant \frac{1}{n}\,.$$

So the given series diverges by comparison with the harmonic series.

(e) This series *converges*. Note that the geometric series $\sum_{n=0}^{\infty} c^n a = \frac{a}{1-c}$ converges since $c \in (0,1)$. Now use #2(f) to compare $\sum_{n=0}^{\infty} |a_n - a|$ with $\sum_{n=0}^{\infty} c^n a$. This shows that the given series converges (in fact, absolutely).