University of Wyoming

Math 3205—Fall 2020

# Analysis I

Department of Mathematics

## HW2

Due Thursday, October 22, 2020

This assignment is graded out of 80, with 15 bonus points available. (The total grade will be capped at 80 points.)

1. (*20 points*) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{b}, & \text{if } x = \frac{a}{b} \in \mathbb{Q} \text{ in lowest terms } (a, b \text{ being relatively prime} \\ & \qquad \text{integers with } b > 0); \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

   (a) Where is $f$ continuous? Justify your answer.

   (b) Compute the upper and lower Riemann sum approximations to $\int_0^1 f(x)\,dx$ using $n = 4$ subintervals, each of width $\frac{1}{n} = \frac{1}{4}$.

   (c) Compute the upper and lower Riemann sum approximations to $\int_0^1 f(x)\,dx$ using $n = 20$ subintervals, each of width $\frac{1}{n} = \frac{1}{20}$.

   (d) Conjecture the value of $\int_0^1 f(x)\,dx$. Give well-reasoned *informal arguments* to support your guess.

2. (*35 points*) When you turn on your calculator (in *radians* mode) it displays 0. After repeatedly pressing the 'cos' button, the display converges to some number. The purpose of this exercise is to explain what is really happening here.

   (a) What is the number displayed on your calculator, when you implement this procedure (after repeating until there is no further visible change in the display)? State the answer to the extent of accuracy of your calculator.

   (b) State a clear recursive definition for the sequence $(a_n)$ expressing the value in the display after the 'cos' button has been pressed $n$ times.

   (c) Using Calculus I, prove that the equation $x = \cos x$ has exactly one real solution. Denote this solution by $a$. (You are *not* required to express $a$ exactly by any simple closed formula.)

   (d) Using the Mean Value Theorem for Derivatives (also from Calculus I), prove that for every real number $x \neq a$, there exists a real number $y$ between $x$ and $a$, such that $\cos x - \cos a = (-\sin y)(x - a)$.

(e) Find an explicit interval containing $a$, and a constant $c \in (0,1)$, such that $|\cos x - a| \leqslant c|x-a|$ for all $x$ in your interval.

(f) Argue by induction that $|a_n - a| \leqslant c^n a$ for every non-negative integer $n$.

(g) Prove that $(a_n) \to a$.

3. (*15 points*) The point of this exercise is to demonstrate the futility of trying to define sums of uncountably many real terms (as stated in the instructional video on uncountability). Let $A = \{a_i : i \in I\}$ be a set of *positive* real numbers indexed by a set $I$. We will assume the sum of *all* elements of $A$ has a meaningful value $\sum_{i\in I} a_i = L \in \mathbb{R}$. From this we will show that $I$ must be countable (i.e. finite or countably infinite). Clearly, we may assume $I$ is infinite. For every sequence $i_1, i_2, i_3, \ldots$ of distinct indices in $I$, we have a series

$$\sum_{j=1}^{\infty} a_{i_j} = a_{i_1} + a_{i_2} + a_{i_3} + \cdots.$$

Now every such series converges, since its partial sums are bounded above by $L = \sum_{i\in I} a_i < \infty$.

(a) Under our assumptions, show that for every positive integer $n$, there are only finitely many indices $i \in I$ for which $a_i > \frac{1}{n}$.

(b) Show that there are only countably many indices in $\bigcup_{n=1}^{\infty}\{i \in I : a_i > \frac{1}{n}\}$.

(c) Conclude that $I$ must be countable.

4. (*25 points*) Determine whether each of the following series converges or diverges. Justify your answers. In addition to techniques learned in our class, you may use any methods learned in Calculus II.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^{10}}{2^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}$

(e) $\displaystyle\sum_{n=0}^{\infty} (a_n - a)$ where $a_n$ and $a$ are as in #2