Name .....



Test

9:35–10:50 am, Thursday October 29, 2026

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an $8.5'' \times 11''$ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Total value of questions: 100 points (plus 8 bonus points).

1. *(18 points)* Consider the sets $A = \{0, 2, 5\}$ and $B = \{2, 5, 6\}$. Evaluate each of the following:

   (a) The power set of $A \cap B$.

   (b) The *number* of subsets of $A \cup B$.

   (c) The *number of subsets* of the *power set* of B.

   (d) The *number* of functions from A to B.

   (e) The *number* of *one-to-one* functions from A to B.

   (f) The *number* of functions from A *onto* B.

2. *(20 points)* Fill in the blanks below using words selected from the following list:

Gödel	onto	converse	contradiction	function	assumption	prove
Hilbert	agrees	disagrees	continuous	bijection	induction	count
Cantor	digits	assume	cardinality	variable	one-to-one	values

to make the following a valid proof. (Some words from this list may be used more than once, and some not at all.)

Theorem (_____). *The set of real numbers is uncountable.*

Proof. The function $g(x) = \frac{e^x}{1+e^x}$ defines a _____ from $\mathbb{R}$ to the open interval $(0,1)$, so $|\mathbb{R}| = |(0,1)|$. So it is enough for us to show that the interval $(0,1)$ is uncountable. We also have a _____ function $h : \mathbb{N} \to (0,1)$ given by $h(n) = \frac{1}{n+1}$, so $|\mathbb{N}| \leqslant |(0,1)|$. So in order to _____ the strict inequality $|\mathbb{N}| < |(0,1)|$, it is enough to _____ that there is no bijection between these two sets. We will argue by _____ : Suppose $f : \mathbb{N} \to (0,1)$ is a bijection. From this assumption, we must obtain a contradiction.

Each of the real numbers $f(n) \in (0,1)$ can be expanded in a decimal expansion with decimal digits $a_{n,1}$, $a_{n,2}$, $a_{n,3}$, ... where $a_{n,k} \in \{0,1,2,3,\ldots,9\}$; this means that $f(n) = \sum_{k=1}^{\infty} \frac{a_{n,k}}{10^k}$. Now consider the real number $b \in (0,1)$ whose decimal _____ $b_1, b_2, b_3, \ldots$ are given by

$$b_k = \begin{cases} 3, & \text{if } a_{k,k} \neq 3; \\ 7, & \text{if } a_{k,k} = 3. \end{cases}$$

Thus $b = \sum_{k=1}^{\infty} \frac{b_k}{10^k} \in (0,1)$. By _____ , f is a bijection; so $b = f(n)$ for some $n \in \mathbb{N}$. However, the n^{th} decimal digit of b is $b_n \neq a_{n,n}$ by our choice of the sequence of digits for b. This _____ with the n^{th} decimal digit of $f(n)$, which is $a_{n,n}$; so $b \neq f(n)$. This is a _____ , and so our theorem is proved. □

3. *(20 points)* Indicate whether each of the following statements is True or False. Here P, Q, R are atomic propositions (each being true or false).

(a) $P \wedge P$ is logically equivalent to P. _____*(True/False)*

(b) $P \vee R$ is logically equivalent to $R \vee P$. _____*(True/False)*

(c) $P \to P$ is a tautology. _____*(True/False)*

(d) $P \to (\neg P)$ is a contradiction. _____*(True/False)*

(e) $(P \wedge (\neg P)) \to Q$ is a tautology. _____*(True/False)*

(f) $P \to Q$ is logically equivalent to $(\neg P) \to (\neg Q)$. _____*(True/False)*

(g) $P \wedge (Q \vee R)$ is logically equivalent to $(P \wedge Q) \vee (P \wedge R)$. _____*(True/False)*

(h) $P \vee (Q \wedge R)$ is logically equivalent to $(P \vee Q) \wedge (P \vee R)$. _____*(True/False)*

(i) $(P \to Q) \to R$ is logially equivalent to $P \to (Q \to R)$. _____*(True/False)*

(j) $(P \to Q) \to (Q \to P)$ is a contradiction. _____*(True/False)*

4. *(20 points)* The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, 34, 55, 89, \ldots$ is defined recursively by

$$F_n = \begin{cases} n, & \text{for all } n \in \{0, 1\}; \text{ and} \\ F_{n-1} + F_{n-2}\,, & \text{for all } n \geqslant 2. \end{cases}$$

The closed formula

$$F_n = \tfrac{1}{\sqrt{5}}(\tau^n - \overline{\tau}^n)$$

for the Fibonacci numbers was proved in class, and you may assume this. Here

$$\tau = \tfrac{1}{2}(1 + \sqrt{5}) \quad \text{and} \quad \overline{\tau} = \tfrac{1}{2}(1 - \sqrt{5}).$$

The related sequence of integers $L_0, L_1, L_2, \ldots$ is defined by $\tau^n = \frac{1}{2}(L_n + F_n\sqrt{5})$ and $\overline{\tau}^n = \frac{1}{2}(L_n - F_n\sqrt{5})$.

From this information, it follows that the sequence $L_0, L_1, L_2, \ldots$ can be generated recursively in a manner very similar to the Fibonacci sequence. Fill in the four missing constants in this recursive definition:

$$L_n = \begin{cases} \square\,, & \text{if } n = 0; \\ \square\,, & \text{if } n = 1; \text{ and} \\ \square\, L_{n-1} + \square\, L_{n-2}\,, & \text{if } n \geqslant 2. \end{cases}$$

In fact, the recurrence formulas for F_n and for L_n may be understood to hold *for all* integers $n \in \mathbb{Z}$, not just for $n \geqslant 2$. Assuming this fact, complete the sixteen missing entries in the following table of values:

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_n						0	1	1	2	3	5
L_n											

5. *(30 points)* Answer TRUE or FALSE to each of the following statements.

(a) Every integer in $\{1.41, 2.39\}$ is greater than 5. _____*(True/False)*

(b) There exists a positive real number x satisfying $0 < x < \frac{1}{n}$ for every positive integer n. _____*(True/False)*

(c) The real number $0.99999999\cdots$ (the decimal number repeating forever) is exactly equal to 1. _____*(True/False)*

(d) The set of all bitstrings of finite length (finite sequences of bits (0's and 1's) is countable. _____*(True/False)*

(e) There exist two real irrational numbers α, β such that both $\alpha+\beta$ and $\alpha\beta$ are irrational. _____*(True/False)*

(f) Every integer is either even or odd (with an exclusive 'or', meaning that no integer is both even and odd). _____*(True/False)*

(g) $(\forall x \in \mathbb{R})(\forall \varepsilon > 0)(\exists a \in \mathbb{Q})(|x - a| < \varepsilon)$ _____*(True/False)*

(h) The equation $x^2 + y^2 = 100$ has infinitely many integer solutions for (x, y). _____*(True/False)*

(i) The equation $x^2 - y^2 = 100$ has infinitely many integer solutions for (x, y). _____*(True/False)*

(j) The equation $x^2 + y^2 = z^2$ has infinitely many integer solutions for (x, y, z). _____*(True/False)*