SOLUTIONS to the Test

October, 2026

1. Please, please, please! (a) asks for a *set*, whereas (b,c,d,e,f) ask for *numbers*.

1. (a) $\mathcal{P}(A \cap B) = \mathcal{P}(\{2,5\}) = \{\varnothing, \{2\}, \{5\}, \{2,5\}\}$.

   (b) $|\mathcal{P}(A \cup B)| = |\mathcal{P}(\{0,2,5,6\}| = 2^4 = 16$.

   (c) $|\mathcal{P}(\mathcal{P}(B))| = 2^{2^3} = 2^8 = 256$.

   (d) $|B^A| = 3^3 = 27$. There are 3 choices for each of $f(0)$, $f(2)$ and $f(5)$.

   (e) $3! = 6$ bijections $A \to B$. Every one-to-one function $A \to B$ is bijective.

   (f) $3! = 6$ bijections $A \to B$. Every function from A onto B is bijective.

2. **Theorem (Cantor).** *The set of real numbers is uncountable.*

   Proof. The function $g(x) = \frac{e^x}{1+e^x}$ defines a **bijection** from $\mathbb{R}$ to the open interval $(0,1)$, so $|\mathbb{R}| = |(0,1)|$. So it is enough for us to show that the interval $(0,1)$ is uncountable. We also have a **one-to-one** function $h : \mathbb{N} \to (0,1)$ given by $h(n) = \frac{1}{n+1}$, so $|\mathbb{N}| \leqslant |(0,1)|$. So in order to **prove** the strict inequality $|\mathbb{N}| < |(0,1)|$, it is enough to **prove** that there is no bijection between these two sets. We will argue by **contradiction**: Suppose $f : \mathbb{N} \to (0,1)$ is a bijection. From this assumption, we must obtain a contradiction.

   Each of the real numbers $f(n) \in (0,1)$ can be expanded in a decimal expansion with decimal digits $a_{n,1}$, $a_{n,2}$, $a_{n,3}$, ... where $a_{n,k} \in \{0,1,2,3,\ldots,9\}$; this means that $f(n) = \sum_{k=1}^{\infty} \frac{a_{n,k}}{10^k}$. Now consider the real number $b \in (0,1)$ whose decimal **digits** $b_1, b_2, b_3, \ldots$ are given by

$$b_k = \begin{cases} 3, & \text{if } a_{k,k} \neq 3; \\ 7, & \text{if } a_{k,k} = 3. \end{cases}$$

   Thus $b = \sum_{k=1}^{\infty} \frac{b_k}{10^k} \in (0,1)$. By **assumption**, f is a bijection; so $b = f(n)$ for some $n \in \mathbb{N}$. However, the n^{th} decimal digit of b is $b_n \neq a_{n,n}$ by our choice of the sequence of digits for b. This **disagrees** with the n^{th} decimal digit of $f(n)$, which is $a_{n,n}$; so $b \neq f(n)$. This is a **contradiction**, and so our theorem is proved. □

3. (a) **T** (b) **T** (c) **T** (d) **F** (e) **T** (f) **F** (g) **T** (h) **T** (i) **F** (j) **F**

4. From the initial values $\tau^n = 1, \frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}$ (for $n = 0, 1, 2$), we immediately read off $L_n = 2, 1, 3$ for $n = 0, 1, 2$. These values, together with the assumption that $L_n = aL_{n-1} + bL_{n-2}$, means that you can directly solve for $a = b = 1$ and so the sequence L_n has a recursive formula

$$L_n = \begin{cases} 2\,, & \text{if } n = 0; \\ 1\,, & \text{if } n = 1; \text{ and} \\ 1L_{n-1} + 1L_{n-2}\,, & \text{if } n \geqslant 2. \end{cases}$$

Note that this sequence (known as the *Lucas numbers*) satisfies the same recurrence relation as the Fibonacci numbers: each term is the sum of the previous two terms. You now have enough information to complete all blanks in the table of values provided:

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
F_n	5	-3	2	-1	1	0	1	1	2	3	5
L_n	-11	7	-4	3	-1	2	1	3	4	7	11

How would one derive the recurrence formula $L_n = L_{n-1} + L_{n-2}$ if you were not given the fact that the Lucas numbers satisfy a second order linear recurrence relation with constant coefficients? The same way as we derived this recurrence relation for the Fibonacci numbers, in class. The roots of $x^2 - x - 1$ are $\frac{1}{2}(1 \pm \sqrt{5})$, i.e. τ and $\overline{\tau}$. Multiplying both sides of $\tau^2 = \tau + 1$ and $\overline{\tau}^2 = \overline{\tau} + 1$ by τ^{n-2} or $\overline{\tau}^{n-2}$ shows that $\tau^n = \tau^{n-1} + \tau^{n-2}$ and $\overline{\tau}^n = \overline{\tau}^{n-1} + \overline{\tau}^{n-2}$. Thus, the sequence of powers of τ (or $\overline{\tau}$) satisfies the same recurrence relation as the Fibonacci numbers: each term is the sum of the previous two. Since $L_n = \tau^n + \overline{\tau}^n$ and $F_n = \frac{1}{\sqrt{5}}(\tau^n - \overline{\tau}^n)$, both L_n and F_n satisfy the same second order recurrence formula. (Given two sequences satisfying the same linear recurrence relation, then any linear combination of those two sequences also satisfies the same linear recurrence formula.)

5. (a) **T** (b) **F** (c) **T** (d) **T** (e) **T** (f) **T** (g) **T** (h) **F** (i) **F** (j) **T**

Some comments and explanations, provided for your benefit only (not required for answering #5):

(a) There is no integer in $\{1.41, 2.39\}$, so the statement is true. (Vacuously so).

(b) The fact that $\mathbb{R}$ does not contain any such 'infinitesimal' value x was emphasized in class. This is known as the Archimedean property of $\mathbb{R}$.

(c) The value of the repeating decimal is the limit of the sequence of approximations 0.9, 0.99, 0.999, ... This value is 1.

(d) As explained in class. List the 2^n bitstrings of length n for each $n \geqslant 0$. This forms one long list with countably many entries.

(e) For example, $\alpha = \sqrt{2}$ and $\beta = 1+\sqrt{2}$.

(f) As explained in class. (The formal proof requires mathematical induction, but the statement should nevertheless be familiar. Just remember that there are no exceptions. Zero is not an exception, and negative integers are not exceptions.)

(g) Given $x \in \mathbb{R}$ and $\varepsilon > 0$, there exists $n > 1$ such that $10^{-n} < \varepsilon$. Round off the decimal expansion of x to n decimal places to get $a \in \mathbb{Q}$ approximating x to within ε.

(h) The equation $x^2 + y^2 = 100$ requires $x, y \in \{0, \pm1, \pm2, \pm3, \ldots, \pm10\}$. This means that only finitely many solutions are possible.

(i) The equation $x^2 - y^2 = (x + y)(x - y) = 100$ requires both $x + y$ and $x - y$ to belong to $\{\pm1, \pm2, \pm4, \pm5, \ldots, \pm50, \pm100\}$, the set of divisors of 100. This means that only finitely many solutions are possible.

(j) For every integer k, the triple $(3k, 4k, 5k)$ is a solution. So is $(k, 0, k)$, and $(0, k, 0)$, and many other triples.