Solutions to Sample Test

October, 2026

1. **Theorem.** *Let S be a finite* **set** *of* **cardinality** *$n = |S|$. Then its* **power** *set has cardinality $|\mathcal{P}(S)| = 2^n$.*

   Proof. We prove the result by **induction** on n. In the **initial** case $n = 0$, the set $S = \varnothing$ is empty; so S has exactly one subset, i.e. $\mathcal{P}(S) = \{\varnothing\}$. Here $|\mathcal{P}(S)| = 1 = 2^0$. The conclusion holds in this case.

   Now let $k \geqslant 0$, and let S be a set with exactly $k+1$ elements. This means that $S = \{a\} \cup T$ for **some** set T of cardinality $|T| = k$ with $a \notin T$. Assuming $|\mathcal{P}(T)| = 2^k$ (the inductive hypothesis), we must **prove** that $|\mathcal{P}(S)| = 2^{k+1}$. Every subset $A \subseteq S$ satisfies exactly one of the following two cases.

   Case (i): $a \notin A$. In this case, of course, $A \subseteq T$. There are exactly $|\mathcal{P}(T)| = 2^k$ subsets A of this form.

   Case (ii): $a \in A$. In this case, of course, $A = \{a\} \cup B$ where $B \subseteq T$ (and in fact, $B = T \setminus \{a\}$). There are exactly $|\mathcal{P}(T)| = 2^k$ subsets B of this form, each of which gives rise to exactly one **set** $A = \{a\} \cup B \subset S$ containing a.

   Combining the possibilities from (i) and (ii), we see that the total **number** of subsets of S is exactly $2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$, as claimed. The statement of he theorem follows by **induction**. □

2. (a) True. P asserts that there exists $x \in \bigcap_{n\in\mathbb{N}}(a_n, b_n)$. By the definition of this intersection, this element x satisfies $x \in (a_m, b_m) \cap (a_n, b_n)$ for all $m, n \in \mathbb{N}$.

   (b) True. Let $m, n \in \mathbb{N}$. Assuming P, $(a_m, b_m) \cap (a_n, b_n)$ is a nonempty open interval, so it contains at least one rational number (actually, infinitely many rational numbers).

   (c) False. A counter example is given by the intervals $(a_n, b_n) = (0, \frac{1}{n})$. Any two of these intervals have non-empty intersection; but the intersection of all these intervals is empty.

   (d) False. Same argument as in (c).

3. (a) No; this comment completely misses the point. Logical arguments can be made about any subject. Certainly one has sets of horses, just as well as one has sets of colors or people or numbers.

(b) Yes. The initial case is not flawed in any essential way, although one must be careful how to interpret the question when a single horse is multicolored.

(c) The statement $P(n)$ is true only for $n = 1$; because for $n \geqslant 2$, one can in principle have a set of horses of different color (the statement of the 'Theorem' nontwithstanding).

(d) The statement $P(n) \to P(n+1)$ is true for $n \geqslant 2$ but false for $n = 1$. This is because the sets A and B have nonempty intersection for $n \geqslant 2$, but empty intersection for $n = 1$ (so no horse $h \in A \cap B$ exists as asserted by the 'Proof').

(e) See (d). In order for the proof by mathematical induction to succeed, we would require the inductive step $P(n) \to P(n+1)$ to hold for every $n \geqslant 1$. In fact, the inductive step fails for $n = 1$.

4. (a) True.

(b) True. In order to simplify the language, let us rephrase the assertion as follows: For every integer $n \geqslant 0$, the terms F_{3n}, F_{3n+1} and F_{3n+2} are even, odd, odd respectively. This is clear for the initial case $n = 0$ by inspecting the starting terms $0, 1, 1$. Now assuming the terms F_{3k}, F_{3k+1} and F_{3k+2} are even, odd, odd respectively, where k is some non-negative integer, then the next three terms have the required parity:

$$F_{3k+3} = F_{3k+2} + F_{3k+1} \text{ is odd} + \text{odd} = \text{even},$$
$$F_{3k+4} = F_{3k+3} + F_{3k+2} \text{ is even} + \text{odd} = \text{odd},$$
$$F_{3k+5} = F_{3k+4} + F_{3k+3} \text{ is odd} + \text{even} = \text{odd}$$

as required.

(c) True. In class we derived the closed formula $F_n = \frac{1}{\sqrt{5}}(\tau^n - \overline{\tau}^n)$ where $\tau = \frac{1}{2}(1+\sqrt{5})$ and $\overline{\tau} = \frac{1}{2}(1-\sqrt{5})$.

(d) True: $|F_n - \frac{1}{\sqrt{5}}\tau^n| = |\frac{1}{\sqrt{5}}\overline{\tau}^n| < 1$. As pointed out in class, F_n is the closest integer value to $\frac{1}{\sqrt{5}}\tau^n$.

(e) False. The number of bitstrings of length n is a power of 2. Most Fibonacci numbers are not powers of 2.

5. (a) T (b) F (c) T (d) T (e) F (f) T (g) F (h) T (i) F (j) F

Some comments and explanations, provided for your benefit only (not required for answering #5):

(a) By Cantor's Theorem, every countably infinite set S has an uncountable power set, i.e. $|\mathcal{P}(S)| > |S|$.

(b) Suppose S is countably infinite. Denote by S^S the set of all functions $S \to S$. We have $|S^S| \geqslant |\mathcal{P}(S)| > |S|$. To prove the first inequality, use the one-to-one function $\mathcal{P}(S) \to S^S$ defined as follows: Fix any two distinct elements $s \neq s'$ in S. Every subset $A \subset S$ gives rise to a function $f_A : S \to S$ defined by $f_A(x) = s$ if

$x \in A$; and $f_A(x) = s'$ if $s \notin A$. It is clear that the map $\mathcal{P}(S) \to S^S$, $A \mapsto f_A$ is one-to-one.

(c) We explained in class why a countable union of countable sets is countable, using the snake-path argument. Construct a table with infinitely many rows and columns, in which the elements of S_i are listed in row i. A snake-path through this table includes each element of $\bigcup_{i=1}^{\infty} S_i$.

(d) Infinite binary sequences are in one-to-one correspondence with subsets of $\mathbb{N}$: the subset $A \subseteq \mathbb{N}$ gives rise to the sequence whose i^{th} entry is 0 or 1 according as $i \notin A$ or $i \in A$.

(e) Quite to the contrary, the MDRP Theorem shows that no such algorithm exists. (I don't expect you to remember the name of the Theorem, but I do expect you would remember me saying several times that no such algorithm can exist.)

(f) This is the statement of the Bernstein-Schröder-Cantor Theorem, which tells us why it is reasonable to compare sets according to their cardinality in the way we would hope.

(g) Not at all. Russell's Paradox says only that we must be careful in our definition of which collections can be called sets.

(h) As emphasized both in class and in the textbook.

(i) $Q \to P$ is the *converse* of $P \to Q$.

(j) The only two subsets of the indicated set are $\varnothing$ and $\{\{\{\{\varnothing\}\}\}\}$ itself. (Recall: If $|S| = n$ then $|\mathcal{P}(S)| = 2^n$. We are given an instance of this formula with $n = 1$.)