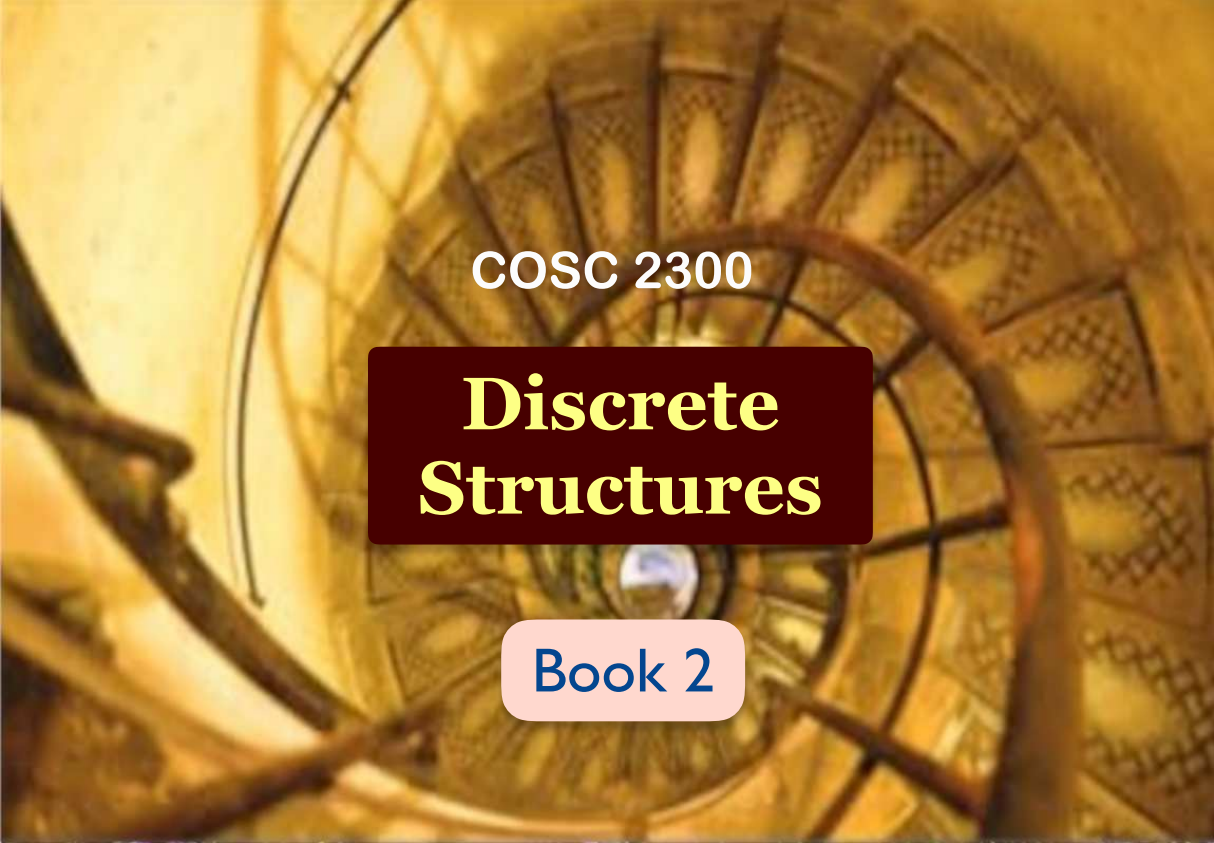




COSC 2300

Discrete Structures

Book 2

If $x, y \in \mathbb{R}$, then the average or mean or arithmetic mean of x and y is $\frac{x+y}{2}$.

Assuming $x, y > 0$, their geometric mean is $\sqrt{xy}$.

eg. the arithmetic mean of 3 and 5 is $\frac{3+5}{2} = 4$; the geometric mean of 3 and 5 is $\sqrt{3 \cdot 5} = \sqrt{15} \approx 3.872983$.

Theorem If $x, y > 0$ then $\underbrace{\sqrt{xy}}_{\text{geometric mean}} \leq \underbrace{\frac{x+y}{2}}_{\text{arithmetic mean}}$. Moreover, equality holds iff $x=y$.

Proof Since $(x-y)^2 \geq 0$ we have $x^2 - 2xy + y^2 \geq 0$ so $x^2 + 2xy + y^2 \geq 4xy$ i.e. $(x+y)^2 \geq 4xy$.
So $x+y \geq \sqrt{4xy} = 2\sqrt{xy}$. Finally, $\frac{x+y}{2} \geq \sqrt{xy}$. $\square$

This is a direct proof... but where did the idea come from?

$$\frac{x+y}{2} \geq \sqrt{xy}$$

$$\Leftrightarrow x+y \geq 2\sqrt{xy}$$

$$\Leftrightarrow (x+y)^2 \geq 4xy$$

$$\Leftrightarrow x^2 + 2xy + y^2 \geq 4xy$$

$$\Leftrightarrow x^2 - 2xy + y^2 \geq 0$$

$$\Leftrightarrow (x-y)^2 \geq 0$$

Does the equation $x^2 + y^2 = 22$ have any integer solution(s)? Justify your answer.

Answer: No. Any integer solution (x, y) can be taken from $x, y \in \{0, 1, 2, 3, 4\}$ without loss of generality. We consider the value of $x^2 + y^2$ for each (x, y) pair in this set:

$$x^2 + y^2 \in \{0, 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 32\}$$

so $x^2 + y^2 \neq 22$.

We used case analysis.

Scratch:

x	0	1	2	3	4
y	0	1	2	3	4
0	0	1	4	9	16
1	1	2	5	10	17
2	4	5	8	13	20
3	9	10	13	18	25
4	16	17	20	25	32

Does the equation $x^2 - y^2 = 22$ have any integer solution(s)? Justify your answer.

Answer: No. We are asking for integer solutions of $(x+y)(x-y) = 22$. Note that

$x+y$ and $x-y$ have the same parity (they are both even or both odd since $(x-y)+2y = x+y$.)

If $x+y$ and $x-y$ are both odd, then $x^2 - y^2 = (x+y)(x-y)$ is odd; but if $x+y$ and $x-y$ are both even, then $(x+y)(x-y)$ is a multiple of 4. However, 22 is neither odd nor a multiple of 4.

Diophantine equations: polynomial equations in several variables for which integer (or rational) solutions are required.

There is no algorithm for answering such problems in general.

If α, β are real irrational, can $\alpha + \beta$ be rational? Justify your answer.

Yes, for example $\underbrace{\pi}_{\text{irrational}} + \underbrace{(4-\pi)}_{\text{irrational}} = 4 \in \mathbb{Q}$.

If you don't know that π is irrational then use this example: $\underbrace{\sqrt{2}}_{\text{irrational}} + \underbrace{(4-\sqrt{2})}_{\text{irrational}} = 4 \in \mathbb{Q}$

If α, β are real irrational, can α^β be rational? Justify your answer.
Yes; for example $\sqrt{2}^{\sqrt{2}} = 2$.

If α, β are real irrational, can α^β be rational? Justify your answer.

If $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ then we're done. Otherwise $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$ and then
 $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$. This proves that such α, β exist.

Scratch:
Try $\sqrt{2}^{\sqrt{2}}$.

This is a nonconstructive proof of existence of such α, β .

Theorem There exist irrational numbers in $\mathbb{R}$.

Proof Clearly $\mathbb{Q} \subseteq \mathbb{R}$ so $|\mathbb{Q}| \leq |\mathbb{R}|$. If $\mathbb{Q} = \mathbb{R}$ then $|\mathbb{Q}| = |\mathbb{R}|$, contrary to Cantor's theorem that $|\mathbb{Q}| < |\mathbb{R}|$ ($\mathbb{Q}$ is countable but $\mathbb{R}$ is not). $\square$

Proof that e, π are irrational: I have proofs on my website ericmoorhouse.org/handouts

Actually π and e are transcendental. (stronger than irrational).

e^π is irrational. π^e ? Nobody knows.

$e+\pi$, $e\pi$? We know that at least one of $e+\pi$ and $e\pi$ is irrational. This is known because:

$$(x-e)(x-\pi) = x^2 - (e+\pi)x + e\pi$$

e and π are transcendental: not roots of any polynomial in x with rational coefficients.

Algebraic numbers: roots of polynomials with rational coefficients.

$\frac{3}{5}$ is a root of $x - \frac{3}{5}$ so $\frac{3}{5}$ is algebraic.

$\sqrt{2}$ is a root of $x^2 - 2$ so $\sqrt{2}$ is algebraic.

There are only countably many algebraic numbers.

Transcendental numbers exist by the same argument as above using cardinality.

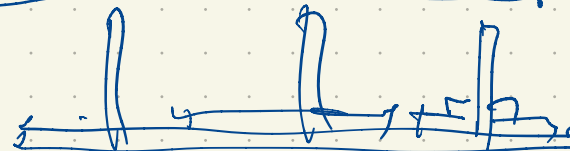
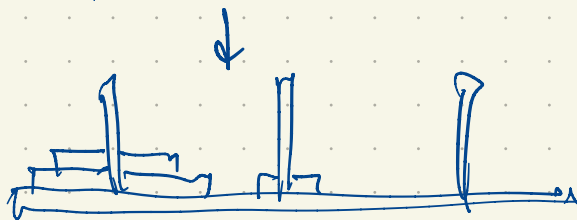
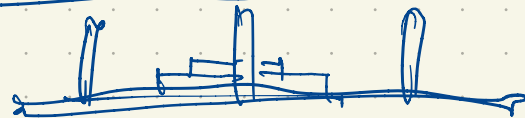
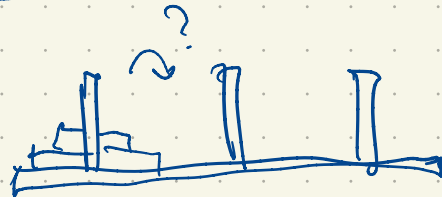
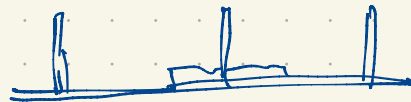
1844 Liouville found a transcendental number.

"Is $x^2 \geq 0$ for any real x ?" "Any" and "every" are often interchangeable. Not always.

For every $x \in \mathbb{R}$, $x^2 \geq 0$. } Universal quantifier
For any $x \in \mathbb{R}$, $x^2 \geq 0$. }
For some $x \in \mathbb{R}$, $x^2 = 0$. } Existential quantifier

Is any real number transcendental? Yes, for example e and π .

Is every real number transcendental? No, for example $\frac{3}{5}$, $\sqrt{2}$ are algebraic.



Mathematical Induction and Recursion

There is an inductive proof that there exists an algorithm to move a tower of n disks from one stack to another. Beyond this, we can prove (recursive) a formula for the number of steps it takes. (also by induction)

Theorem The Towers of Hanoi puzzle has a solution for any number of disks.

Proof (Initial case) For one disk, simply move the disk from post A to post B.

(General case with $n \geq 2$ disks) Using the previous strategy, we can move the top $n-1$ disks from post A to post C, leaving the largest disk on post A. Then move the

largest disk from post A to post B.

Then move the smallest $n-1$ disks from post C to post B,

again using the strategy

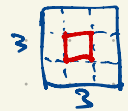
for $n-1$ disks. At this point all disks are stacked on post B in the correct order.

The result of the theorem follows for every $n \geq 1$. $\square$

This is an example of a proof by induction. See the handout on the course website.



Another example: a tiling problem using 2×2 squares with a 1×1 corner removed. I'll call this shape an L-tile.



The 3×3 square cannot be tiled using (three) L-tiles.

Proof In any tiling of the 3×3 square, we consider how the middle 1×1 square is covered.

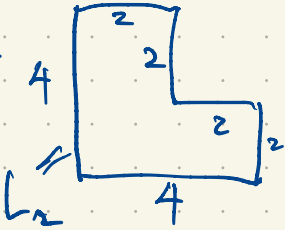
If this is covered by the middle of an L-tile  then the remaining corner is disconnected from the

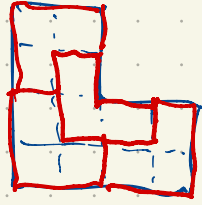
rest and so it cannot complete to a tiling of the entire 3×3 square. Otherwise, the middle 1×1 square is covered by an "end" of some L-tile

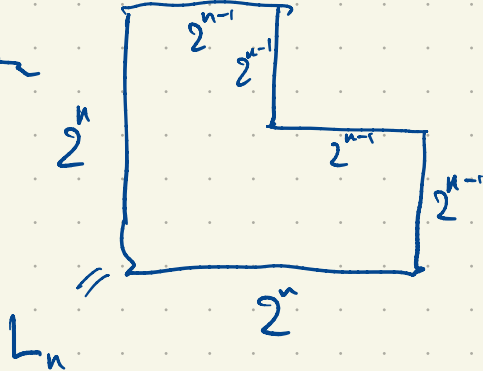
(and without loss of generality the tile is oriented as shown).



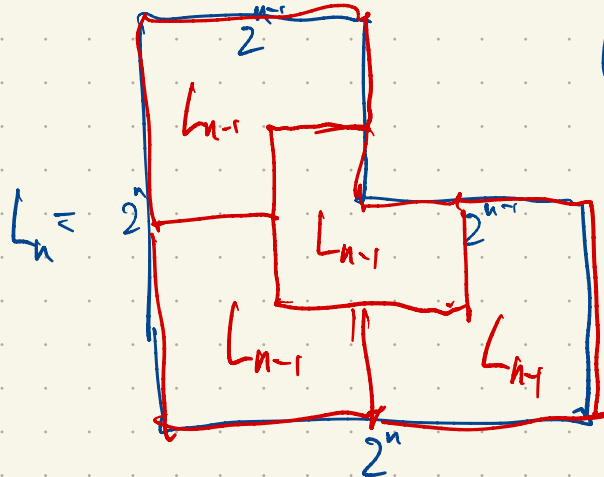
But this leaves another 1×1 square (as shown) that cannot be covered by any L-tile. This proves (by contradiction) that the 3×3 square cannot be tiled using L-tiles. $\square$

The region  can be tiled using L-tiles, $L = 2 \begin{matrix} 1 & 1 \\ 2 & 2 \end{matrix} = L_1$



Theorem The region  can be tiled using L-tiles, for every $n \geq 1$.

Proof We prove the result by induction. For $n=1$, the result is trivial: a single L-tile tiles L_1 . Now if $n \geq 2$ then we divide L_n into four copies of L_{n-1} as shown:



Use the previous step (called the inductive hypothesis) to tile each of the four L_{n-1} -regions using L-tiles; so the entire L_n -region can be tiled using L-tiles.



Formalizing the concept of proof by induction:
 Our goal is to prove $(\forall n \geq 1) (P(n))$ i.e. we want to prove
 $P(1), P(2), P(3), P(4), \dots$

This can be accomplished by proving

- $P(1)$ (the initial case); and

- $(\forall n \geq 1) (P(n) \rightarrow P(n+1))$ ~

the inductive step: $P(n)$ is the ^(inductive) hypothesis of this conditional statement

Recall: Bitstrings (words of 0's and 1's)

" '0', '1', '00', '01', '10', '11', ...

There are 2^n bitstrings of length n .

Let a_n be the number of bitstrings of length n having no two consecutive 1's
 (strings having no substring '11', i.e. 11-free strings)

n	strings	a_n
0	''	1
1	0, 1	2
2	00, 01, 10	3
3	000, 100, 010, 001, 101	5
4	0000, 1000, 0100, 0010, 0001, 1010, 1001, 0101	8

8

13

⋮

5
⋮

Theorem The sequence $a_0, a_1, a_2, a_3, \dots$ is defined recursively by

$$a_n = \begin{cases} 1, & \text{if } n=0; \\ 2, & \text{if } n=1; \\ a_{n-1} + a_{n-2}, & \text{if } n \geq 2. \end{cases}$$

This is a recursive formula for a_n .

The formula $a_n = a_{n-1} + a_{n-2}$ is called a recurrence relation