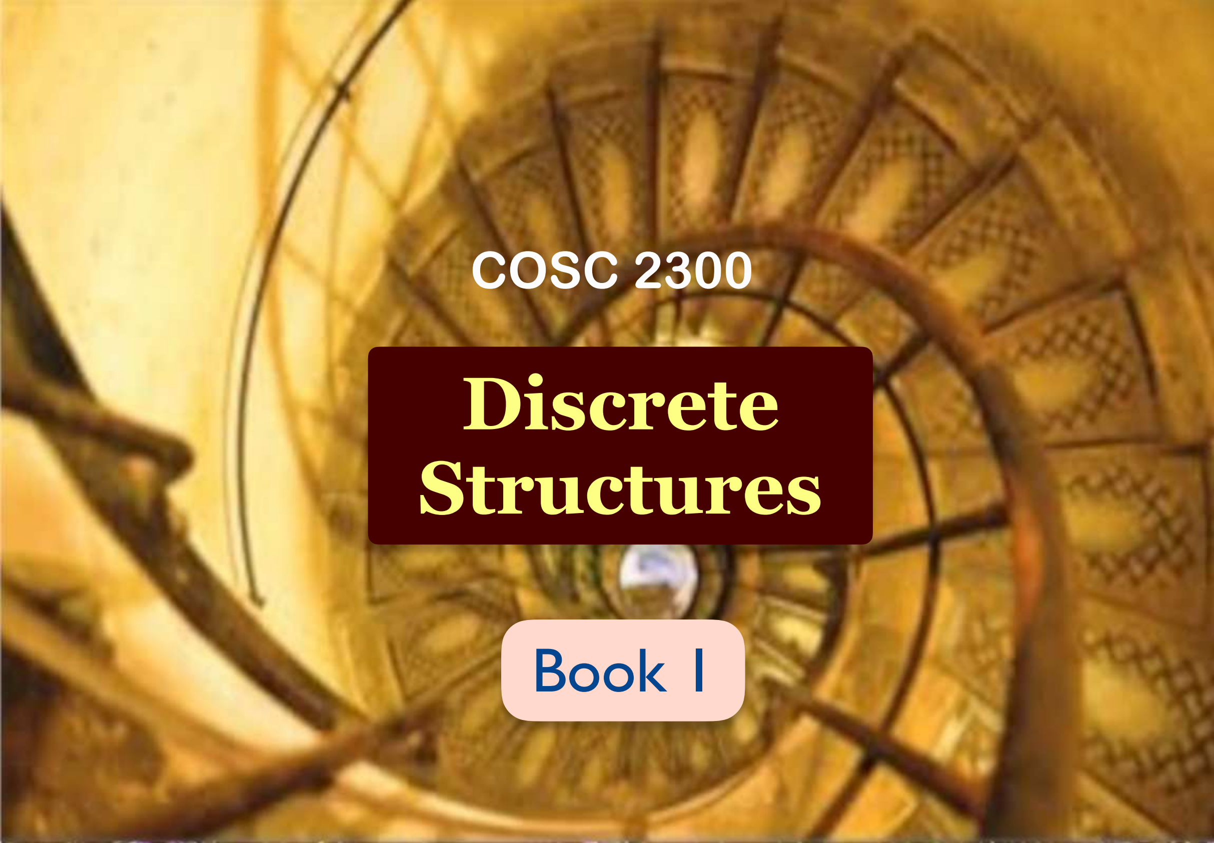
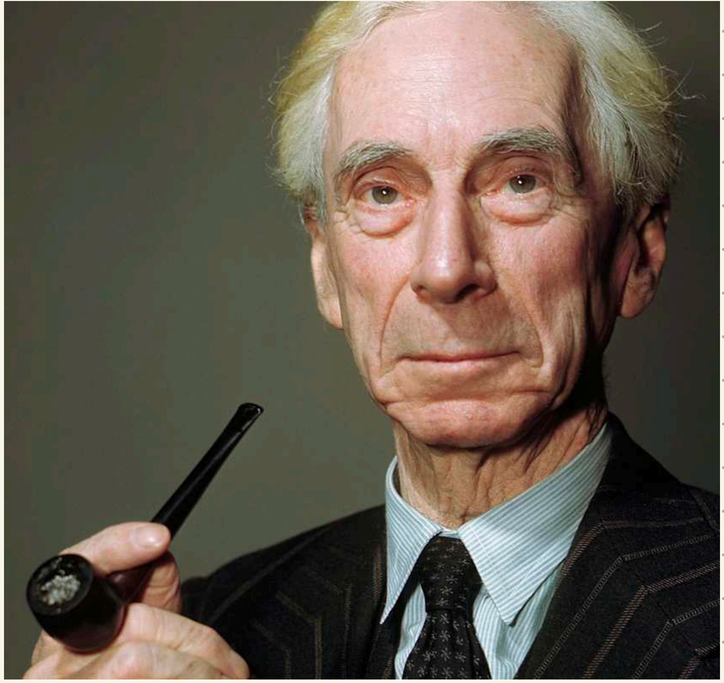


COSC 2300

Discrete Structures

Book 1



Bertrand Russell (1872-1970)

In a particular village there is a gentleman barber who shaves every gentleman who does not shave himself.

Does the barber shave himself?

No true/false answer can be given that is consistent.

This is an example of Russell's Paradox.

Other forms:

P: The statement P (ie. this statement) is false.

Avoiding self-referential statements:

A: The statement B is true.

B: The statement A is false.

Propositional Logic (Propositional Calculus)

R: Bertrand Russell was born in 1872. (True)

M: Mars is the second planet from the sun. (False)

J: January is the first month of the calendar year. (True)

T: A triangle has four sides. (False)

Syntactic vs. semantic features language

syntax (symbolic)

meaning (semantics) true/false

R: Bertrand Russell was born in 1872. (True)
M: Mars is the second planet from the sun. (False)
J: January is the first month of the calendar year. (True)
T: A triangle has four sides. (False)

"Exclusive or" of T and T is F. Avoid!

$R \vee M$ is True
"or"
"join"
"disjunction"

$R \wedge M$ is False
"and"
"meet"
"conjunction"

"inclusive" or

Truth Table: for two staments P, Q
(atomic propositions)

P	Q	$P \vee Q$	$P \wedge Q$
T	T	T	T
T	F	T	F
F	T	T	F
F	F	F	F

P	$\neg P$
T	F
F	T

Conditional Statements $P \to Q$ ("P implies Q")
(implication) ("If P then Q")

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

$R \to M$ is false
$J \to R$ is true.
$T \to M$ is true.

$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$
equivalence of propositions

$P \wedge Q \equiv Q \wedge P$

$P \to Q \equiv (\neg P) \vee Q$

P	Q	$P \to Q$	$\neg P$	$(\neg P) \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$(P\vee Q)\vee R \equiv P\vee(Q\vee R)$

$a(b+c) = ab+ac$

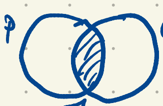
$a+bc \neq (a+b)(a+c)$

$P\vee(Q\wedge R) \equiv (P\vee Q)\wedge(P\vee R)$

$P\wedge(Q\vee R) \equiv (P\wedge Q)\vee(P\wedge R)$

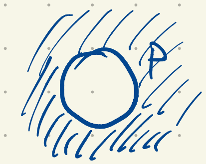
This can be verified using a truth table (with eight rows) or using Venn diagram

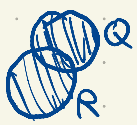
$P\wedge Q$



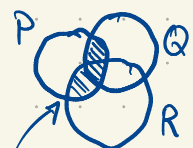
$P\vee Q$

$\neg P$

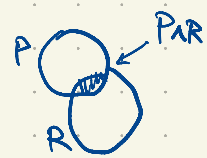


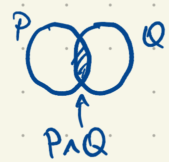


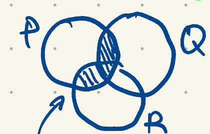
$Q\vee R$



$P\wedge(Q\vee R)$



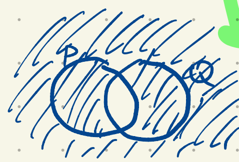




$(P\wedge Q)\vee(P\wedge R)$

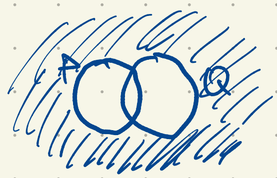
$\neg(P\wedge Q) \equiv (\neg P)\vee(\neg Q)$

"P and Q are not both true, i.e. at least one of P and Q fails"



$\neg(P\vee Q) \equiv (\neg P)\wedge(\neg Q)$

" Neither P nor Q is true; both are false".



De Morgan's Rules

$\neg(P_1\wedge P_2\wedge P_3\wedge\cdots\wedge P_k) \equiv (\neg P_1)\vee(\neg P_2)\vee\cdots\vee(\neg P_k)$

P_i : the i^{th} student is wearing red

$P_1\wedge P_2\wedge\cdots\wedge P_k$: all k students are wearing red

$\neg(P_1\wedge P_2\wedge\cdots\wedge P_k)$: not all k students are wearing red i.e. at least one of the k students is not wearing red.

$\neg(P_1\vee P_2\vee\cdots\vee P_k) \equiv (\neg P_1)\wedge(\neg P_2)\wedge\cdots\wedge(\neg P_k)$: none of the k students are wearing red.

"All the students are not wearing red." Ambiguous! Which of the above does the person mean?

$P \vee (\neg P)$

P	$\neg P$	$P \vee (\neg P)$
T	F	T
F	T	T

$P \vee (\neg P)$ is a <u>tautology</u>: its truth value is always true regardless of P.

P: "There exist integers x, y, z such that $x^3 + y^3 + z^3 = 42$."

$P \vee (\neg P)$ says that P is either true or false.

$P \wedge (\neg P)$ is a <u>contradiction</u>

P	$\neg P$	$P \wedge (\neg P)$
T	F	F
F	T	F

$P \wedge (Q \vee R)$ is neither a tautology nor a contradiction: it depends on P, Q, R (it is a <u>contingent</u> proposition).

$P \to (Q \wedge (\neg Q))$ cannot be true unless P is false.

$P \to (Q \wedge (\neg Q)) \equiv \neg P$.

$P \to (Q \vee (\neg Q))$ is a tautology.

$P \to Q$

Conditional statement

"If P then Q"

or "P implies Q"

"Q whenever P"

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

$P \to Q$ is equivalent to $(\neg P) \vee Q$

$P \to Q \equiv (\neg P) \vee Q$

Biconditional statement $P \leftrightarrow Q$

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$.

"x is even" $\leftrightarrow$ "x^2 is even" This is a true statement for integers.

"P if and only if Q"

"P iff Q"

The conditional statement $Q \to P$ is the converse of $P \to Q$.

In order to prove $P \to Q$, one typically does so by assuming P and then proceeding to prove Q as a consequence.

In order to prove $P \leftrightarrow Q$, one typically proves $P \to Q$ and then also prove $Q \to P$.

Sets e.g. the set $A = \{1, 2, 5\}$ contains three elements (members) $1, 2, 5$:

$1 \in A$
$2 \in A$
$5 \in A$
$7 \notin A \;\equiv\; \neg(7 \in A)$

\in

$2 \in A$

ε epsilon

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \cdots\}$

$\{\,\} = \emptyset$ \varnothing

the empty set

$\{1, 1, 3\} = \{1, 3\}$
$= \{3, 1\}$

$B = \{1, 4, \{1,7\}, \{1,2,4\}, \{2\}\}$ has 5 elements

$1 \in B$
$2 \notin B$
$\{2\} \in B$
$7 \notin B$
$13 \notin B$
$1 \in \{1,7\} \in B$.

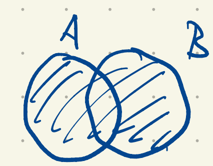
A set is completely specified by its elements.

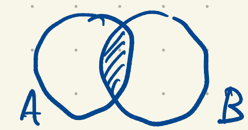
<u>Discrete</u> vs. <u>Continuous</u>

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ $\qquad \mathbb{R}$ $\qquad [5,8] = \{x \in \mathbb{R} \mid 5 \le x \le 8\}$

If A and B are sets then their union is $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ (or: inclusive)

intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

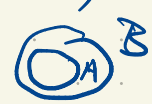




$A \subseteq B$ means $x \in B$ for all $x \in A$. (Every element of A is an element of B.)

"A is a subset of B"

If $A = B$ then $A \subseteq B$. (Not conversely!)



$A = B$ is equivalent to $A \subseteq B$ and $B \subseteq A$.

$\subsetneq \quad \subsetneqq \quad \not\subseteq$

Warning: $A \subset B$ can mean "A is a proper subset of B" (A is a subset of B but not equal to B), but some people use it to just mean "A is a subset of B". Is there any confusion between $x < y$ and $x \le y$?

$\{\ \} = \emptyset$

$\mathcal{P}(A)$

The <u>power set</u> of a set A is the set of all subsets of A:

$$\mathcal{P}(A) = \{B \mid B \subseteq A\}$$

$$\mathcal{P}(\{2,5\}) = \{\emptyset, \{2\}, \{5\}, \{2,5\}\}$$

$$\{2\} \subseteq \{2,5\}$$
$$\{2\} \notin \{2,5\}$$
$$2 \in \{2,5\}$$
$$\{2\} \in \mathcal{P}(\{2,5\})$$

$$\begin{tikzcd}
 & \{2,5\} \arrow[dl, dash] \arrow[dr, dash] & \\
\{2\} \arrow[dr, dash] & & \{5\} \arrow[dl, dash] \\
 & \emptyset = \{\ \} &
\end{tikzcd}$$

If S has n elements then $\mathcal{P}(S)$ has 2^n elements

Can a set be an element of itself?

$\emptyset = \{\ \}$ has no elements

$\mathcal{P}(\emptyset) = \{\emptyset\} = \{\{\ \}\}$: 1 element

$A = \{\emptyset\}$ has one element

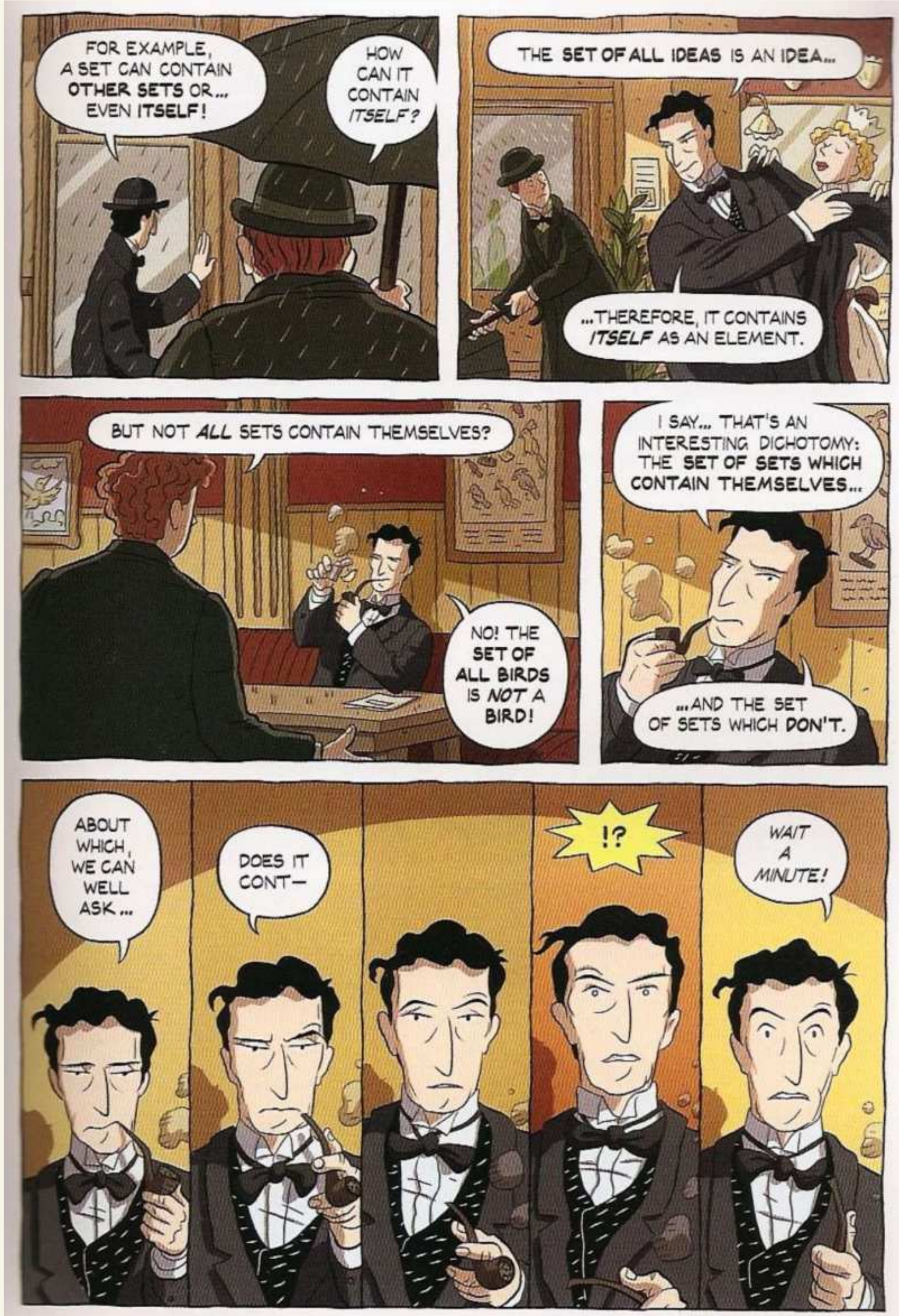
$\Rightarrow \mathcal{P}(A) = \{\emptyset, A\} = \{\emptyset, \{\emptyset\}\} = \{\{\ \}, \{\{\ \}\}\}$

$$\begin{tikzcd}
A \arrow[d, dash] \\
\emptyset
\end{tikzcd}$$

Let $S = \{x \mid x \text{ is a set}, x \notin x\}$

Does $S \in S$? If $S \in S$ then $S \notin S$.
If $S \notin S$ then $S \in S$.

<u>Russell's Paradox</u>



FOR EXAMPLE, A SET CAN CONTAIN OTHER SETS OR... EVEN ITSELF!
HOW CAN IT CONTAIN ITSELF?
THE SET OF ALL IDEAS IS AN IDEA...
...THEREFORE, IT CONTAINS ITSELF AS AN ELEMENT.
BUT NOT ALL SETS CONTAIN THEMSELVES?
NO! THE SET OF ALL BIRDS IS NOT A BIRD!
I SAY... THAT'S AN INTERESTING DICHOTOMY: THE SET OF SETS WHICH CONTAIN THEMSELVES...
...AND THE SET OF SETS WHICH DON'T.
ABOUT WHICH, WE CAN WELL ASK...
DOES IT CONT—
!?
WAIT A MINUTE!

An alphabet is a (usually finite) set of symbols

eg. $\{a, b, c, \cdots, z\}$ (lower case Roman letters)

$\{0, 1\}$ (binary alphabet).

$\{0, 1, 2, \cdots, 9, A, B, C, D, E, F\}$ (hexadecimal alphabet)

ASCII

$A = \{a, b, c, \cdots, z\}$ $|A| = 26$

A word of length n over A is a string of letters, eg. 'dqqjl' is a word (string) of length 5.

A^n = set of all words of length n over the alphabet A. $|A^n| = |A|^n = 26^n$

There are 26^n words of length n over A.

$a_0 + a_1 + a_2 + \cdots = \sum_{n=0}^{\infty} a_n$

$A^0 = \{\text{''}\}$

$A^* = A^0 \cup A^1 \cup A^2 \cup A^3 \cup \cdots = \bigcup_{n=0}^{\infty} A^n$

$= \{\underbrace{\text{''}}_{A^0}, \underbrace{\text{'a'}, \text{'b'}, \text{'c'}, \cdots, \text{'z'}}_{A^1}, \underbrace{aa, ab, \cdots, az, ba, bb, \cdots, bz, \cdots, za, zb, \cdots, zz}_{A^2}, \quad \cdots \}$

The union of three sets A, B, C can be written as $A \cup B \cup C$

If $A \cap B = \emptyset$ and $A \cap C = \emptyset$ and $B \cap C = \emptyset$ then A, B, C are disjoint sets

and $|A \cup B \cup C| = |A| + |B| + |C|$

$|A|$ = size of A = number of elements of A = cardinality of A.

Example: Truth table for $(P\wedge Q)\to R$ and $P\wedge(Q\to R)$

P	Q	R	$P\wedge Q$	$(P\wedge Q)\to R$	$Q\to R$	$P\wedge(Q\to R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	F	T	T	F
F	T	F	F	T	F	F
F	F	T	F	T	T	F
F	F	F	F	T	T	F

Cardinality of sets:

$|A| = |B|$ means there is a one-to-one correspondence between the two sets.

$|A| \leq |B|$ means there is a one-to-one correspondence between elements of A and elements of a subset of B.