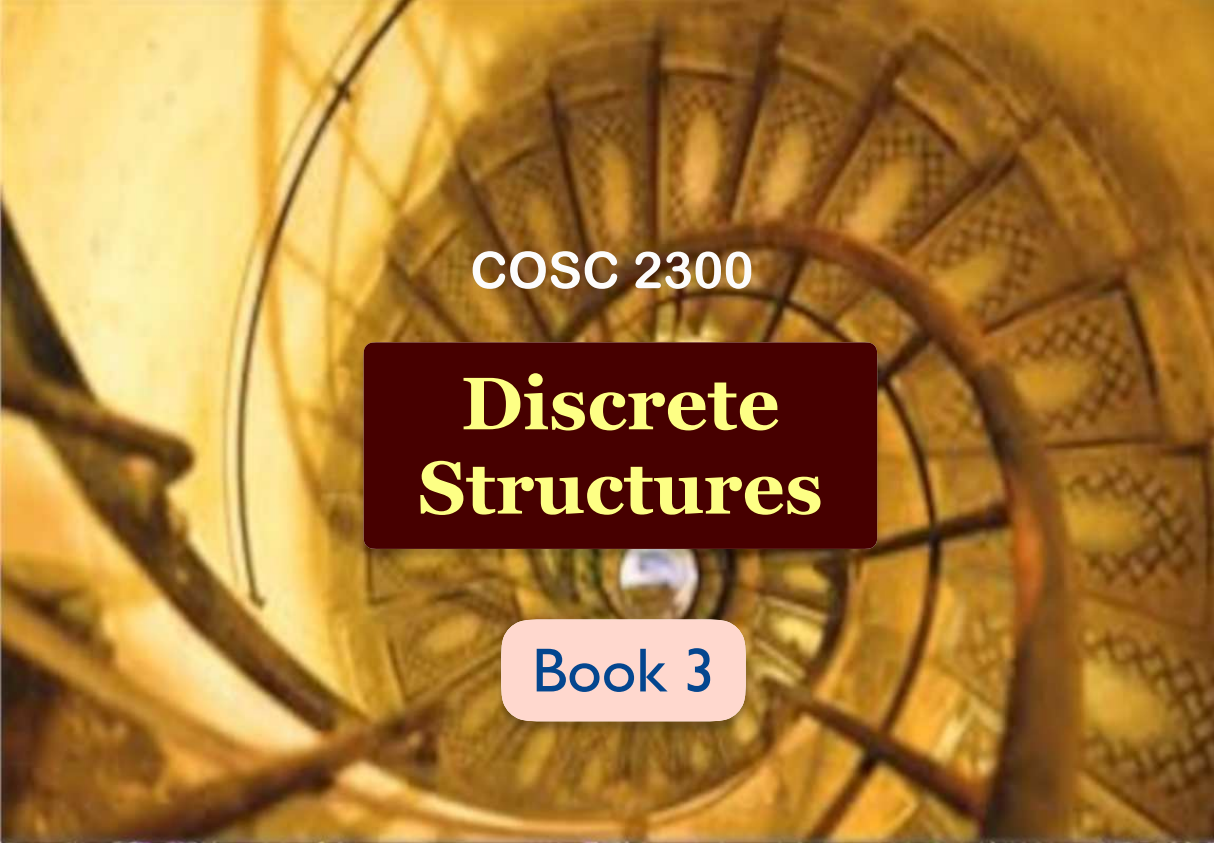

COSC 2300
Discrete Structures
Book 3

I want to give 19 identical silver dollars to five people.

Represent silver dollars by 19 *'s and separate them using 4 |'s eg. ***|*****|**||*********

Answer: $\binom{23}{19} = \binom{23}{4} = 8855$.

What if instead we require each of the five people to get at least one of the silver dollars?
First give out five silver dollars, one apiece, then $\binom{14+4}{4} = 3060$.

$\sum_{k=0}^{n} \binom{n}{k} = 2^n$ First proof: $(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k}$

Second proof: Count subsets of an n-set.

$1 - 6 + 15 - 20 + 15 - 6 + 1 = 0$

$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 0 & \text{for } n \geq 1 \\ 1 & \text{for } n = 0. \end{cases}$ Proof: Use $x=1$, $y=-1$ in the Binomial Theorem.

$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

Eg. $n=3$. $1^2 + 3^2 + 3^2 + 1^2 = 20 = \binom{6}{3} = \frac{6\times5\times4}{3\times2\times1}$

If a, b are integers, we say c divides b if b is a multiple of c

(All variables here are integers)

$\Leftrightarrow$ $b = kc$ for some k

$\Leftrightarrow$ c is a divisor of b

$\Leftrightarrow$ $c \mid b$

The divisors of 12 are $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$. (There are 12 divisors of 12.)
The divisors of 30 are $\pm1, \pm2, \pm3, \pm5, \pm6, \pm10, \pm15, \pm30$. (There are 16 divisors of 30.)
The divisors of 7 are $\pm1, \pm7$. A number $p > 1$ is prime if its only positive divisors are 1 and p.
eg. 7 is prime; 1 is not.

The divisors of 12 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$.

The divisors of 30 are $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30$.

The common divisors of 12 and 30 are $\pm 1, \pm 2, \pm 3, \pm 6$. (There are 8 common divisors of 12 and 30.)

The greatest common divisor of 12 and 30 is 6: $\gcd(12, 30) = 6$.

Computing the gcd of two integers a, b is very practical and is much easier than factoring (if a, b are large).

$\gcd(12, 30) = \gcd(2^2 \cdot 3, 2 \cdot 3 \cdot 5) = 2 \cdot 3 = 6$.

This method is not practical for numbers that are hundreds of digits long.

Eg. $\gcd(124, 46) = \gcd(46, 32) = \gcd(32, 14) = \gcd(14, 4) = \gcd(4, 2) = \gcd(2, 0)$

$124 = 2 \times 46 + 32$
$46 = 1 \times 32 + 14$
$32 = 2 \times 14 + 4$
$14 = 3 \times 4 + 2$
$4 = 2 \times 2 + 0$

The last nonzero remainder here is $\gcd(124, 46) = 2$.

Divisors of 2: $\pm 1, \pm 2$

Divisors of 0: $0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \cdots$

$0 = 0 \cdot 5$

Common divisors of 0 and 2: $\pm 1, \pm 2$

$\gcd(0, 2) = 2$.

This is Euclid's Algorithm for computing $\gcd(a, b)$.

It relies on repeated application of the Division Algorithm:

Given any integers n, d with $d \geq 1$, there exist unique q, r such that

$n = qd + r, \quad r \in \{0, 1, 2, \cdots, d-1\}$

$r = n \bmod d$

q is the quotient;
r is the remainder.