

SAMPLE Test

October, 2026

This sample test is intended to resemble the upcoming Test (9:35–10:50 am, Thursday, October 29 in our usual classroom, CR 314) in approximate length, difficulty, and style, although clearly the content may differ. The actual content will consist of material covered in class and in the related sections of the Textbook.

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an $8.5'' \times 11''$ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Total value of questions: 100 points (plus 3 bonus points).

1. *(20 points)* Fill in the blanks below using words selected from the following list:

set	then	power	implies	number	elements	hypothesis
and	some	empty	equals	because	contains	induction
all	prove	every	initial	greater	therefore	cardinality

to make the following a valid proof. (Some words from this list may be used more than once, and some not at all.)

Theorem. *Let S be a finite* _____ *of* _____ *$n = |S|$. Then its* _____ *set has cardinality $|\mathcal{P}(S)| = 2^n$.*

Proof. We prove the result by _____ on n. In the _____ case $n = 0$, the set $S = \varnothing$ is empty; so S has exactly one subset, with $\mathcal{P}(S) = \{\varnothing\}$. Here $|\mathcal{P}(S)| = 1 = 2^0$. The conclusion holds in this case.

Now let $k \geqslant 0$, and let S be a set with exactly $k+1$ elements. This means that $S = \{a\} \cup T$ for _____ set T of cardinality $|T| = k$ with $a \notin T$. Assuming that $|\mathcal{P}(T)| = 2^k$, we must _____ that $|\mathcal{P}(S)| = 2^{k+1}$. Every subset $A \subseteq S$ satisfies exactly one of the following two cases.

Case (i): $a \notin A$. In this case, of course, $A \subseteq T$. There are exactly $|\mathcal{P}(T)| = 2^k$ subsets A of this form.

Case (ii): $a \in A$. In this case, clearly $A = \{a\} \cup B$ where $B \subseteq T$ (and in fact, $B = T \setminus \{a\}$). There are exactly $|\mathcal{P}(T)| = 2^k$ subsets B of this form, each of which gives rise to exactly one $A = \{a\} \cup B \subset S$ containing a.

Combining the possibilities from (i) and (ii), we see that the total of subsets of S is exactly $2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$, as claimed. The statement of the theorem follows by . □

2. *(18 points)* Consider a collection of nonempty open intervals $(a_n, b_n) \subset \mathbb{R}$ for $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Let P be the statement that $\bigcap_{n\in\mathbb{N}}(a_n, b_n) \neq \varnothing$. Determine whether each of the following statements is TRUE or FALSE. (No proofs are required.)

   (a) $P \to (\forall m \in \mathbb{N})(\forall n \in \mathbb{N})(\exists x \in \mathbb{R})(x \in (a_m, b_m) \cap (a_n, b_n))$

   (b) $P \to (\forall m \in \mathbb{N})(\forall n \in \mathbb{N})(\exists x \in \mathbb{Q})(x \in (a_m, b_m) \cap (a_n, b_n))$

   (c) The converse of (a), i.e. $(\forall m \in \mathbb{N})(\forall n \in \mathbb{N})(\exists x \in \mathbb{R})(x \in (a_m, b_m) \cap (a_n, b_n)) \to P$.

   (d) The converse of (b), i.e. $(\forall m \in \mathbb{N})(\forall n \in \mathbb{N})(\exists x \in \mathbb{Q})(x \in (a_m, b_m) \cap (a_n, b_n)) \to P$.

   (e) $P \leftrightarrow \bigcup_{n\in\mathbb{N}}(a_n, b_n) = \mathbb{R}$.

   (f) $P \leftrightarrow \bigcup_{n\in\mathbb{N}} \overline{(a_n, b_n)} = \mathbb{R}$ where $\overline{(a_n, b_n)} = (-\infty, a_n] \cup [b_n, \infty)$.

3. *(20 points)* Consider the following incorrect 'Theorem' and 'Proof'.

   'Theorem'. All horses have the same color.

   'Proof'. If $n \geqslant 1$ and S is any set of n horses, we show that all horses in S have the same color. The statement is obviously correct for $n = 1$ since any horse has the same color as itself.

   Now consider the case $n = k + 1$ where k is a large positive integer, and let S be any set of $n = k+1$ horses. We may suppose $S = A \cup B$ where A is a set of k horses, and B is a different set of k horses. By the inductive hypothesis, all horses in A have the same color; also, all horses in B have the same color. The subset $A \subset S$ has only one element outside B, and the subset $B \subset S$ has only one element outside A. Take any element (horse) $h \in A \cap B$. Then every horse in A has the same color as h; and every horse in B has the same color as h. This shows that all the horses in S have the same color as h, as claimed. □

   Answer the following:

   (a) Is it fair to say that the fallacy of the 'Theorem' and 'Proof' given lies in the fact that theorems apply only to numbers and not to horses?

   (b) Is the proof given in the initial case at least correct?

(c) Consider the statement $P(n)$: 'Whenever S is a set of n horses, all horses in S have the same color.' For which values of n is $P(n)$ actually true?

(d) For which values of n is the statement $P(n) \to P(n+1)$ true?

(e) Explain the fallacy in the 'Proof' given.

4. *(15 points)* The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, 34, 55, 89, \ldots$ is defined recursively by $F_n = n$ for $n \in \{0, 1\}$; and $F_{n+2} = F_{n+1} + F_n$ for all $n \geqslant 0$. Indicate whether each of the following statements is True or False (you are *not* required to supply any proofs).

(a) The parity of the Fibonacci numbers follows the repeating pattern: even, odd, odd, even, odd, odd, even, odd, odd, etc. (repeating forever).

(b) The property stated in (a) can be proved by mathematical induction.

(c) There is a closed formula for F_n; and this formula can be proved by mathematical induction.

(d) There exist positive constants N, c, r such that for all $n > N$, $|F_n - cr^n| < 1$.

(e) For all $n \geqslant 1$, the number of bitstrings of length n is F_{n+2}.

5. *(30 points)* Answer TRUE or FALSE to each of the following statements.

(a) Every countably infinite set has an uncountable number of subsets.
_____*(True/False)*

(b) If a set S is countable, then the number of functions $S \to S$ is countable.
_____*(True/False)*

(c) If $S_1, S_2, S_3, \ldots$ are countable sets, then the set $S_1 \cup S_2 \cup S_3 \cup \cdots$ is countable.
_____*(True/False)*

(d) The number of infinite sequences of bits (0's and 1's) is uncountable.
_____*(True/False)*

(e) Using the MDRP Algorithm, given any polynomial equation in n variables $x_1, x_2, \ldots, x_n$ with integer coefficients, we can effectively determine whether the equation has at least one integer solution. _____*(True/False)*

(f) Given any sets A and B, if there is a one-to-one function $A \to B$ and a one-to-one function $B \to A$, then there exists a bijection $A \to B$. _____*(True/False)*

(g) Russell's Paradox shows that we cannot reasonably discuss cardinalities for infinite sets. _____*(True/False)*

(h) The cardinality $|\overline{S}|$ of the complement of a set S depends heavily on our understanding of what we regard to be the 'universal set'. _____*(True/False)*

(i) The conditional statement $P \to Q$ is the contrapositive of the conditional statement $Q \to P$. _____*(True/False)*

(j) The set $\{\{\{\{\varnothing\}\}\}\}$ has exactly five subsets. _____*(True/False)*