SOLUTIONS to Test

November 2023

1. Find a pair of linearly independent vectors $\{\mathbf{u}, \mathbf{v}\}$ spanning the plane $4x+5y+7z = 0$ in $\mathbb{R}^3$.

Any pair of vectors in the plane will do, as long as one vector is not a scalar multiple of the other; for example

$$\mathbf{u} = \begin{bmatrix} 7 \\ 0 \\ -4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 7 \\ -5 \end{bmatrix}.$$

A more systematic approach would be to solve the linear equation (as a system of one equation in three unknowns x, y, z), although this approach is probably overkill for such a straightforward problem. The augmented matrix for the system is $\big[4\ 5\ 7 \mid 0\big] \sim \big[1\ \frac{5}{4}\ \frac{7}{4} \mid 0\big]$ in reduced row echelon form. Note that x is a basic variable and y, z are free variables. The general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{5}{4}s-\frac{7}{4}t \\ s \\ t \end{bmatrix} = s\mathbf{u}' + t\mathbf{v}' \quad \text{where } \mathbf{u}' = \begin{bmatrix} -5/4 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{v}' = \begin{bmatrix} -7/4 \\ 0 \\ 1 \end{bmatrix}.$$

This alternative basis $\mathbf{u}', \mathbf{v}'$ has the advantage of being found systematically by our general algorithm; but with the disadvantage of requiring ugly fractions. It is related to the previous basis $\mathbf{u}, \mathbf{v}$ by $\mathbf{u}' = -\frac{5}{28}\mathbf{u} + \frac{1}{7}\mathbf{v}$, $\mathbf{v}' = -\frac{7}{4}\mathbf{v}$.

2. Consider the three vectors

$$\mathbf{v}_1 = \begin{bmatrix} 7 \\ -10 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$$

in $\mathbb{R}^3$. Express the zero vector $\mathbf{0}$ as a nontrivial linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. (This shows that the three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.)

Ouch! I left out a minus sign in the third vector, which meant that the three given vectors were linearly independent (you can see this from the 3×3 matrix having $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ as columns, which has determinant equal to -102. This required me to give everyone full credit on the problem.

Just as in #1, I will describe a faster, inspired way and also a slower, methodical way. By inspection we see that

$$\mathbf{v}_2+2\mathbf{v}_3 = \begin{bmatrix} 0 \\ 9 \\ 11 \end{bmatrix}, \quad \mathbf{v}_1+7\mathbf{v}_3 = \begin{bmatrix} 0 \\ 18 \\ 22 \end{bmatrix} = 2(\mathbf{v}_2+2\mathbf{v}_3),$$

from which $\mathbf{v}_1 - 2\mathbf{v}_2 + 3\mathbf{v}_3 = \mathbf{0}$ is the desired relation.

The more systematic approach would be to solve $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = \mathbf{0}$ as a linear system of three equations in three unknowns a, b, c. Since this system is homogeneous, we are just looking for null vectors of the matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, thus:

$$\begin{bmatrix} 7 & 2 & -1 \\ -10 & 1 & 4 \\ 1 & 5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 \\ -10 & 1 & 4 \\ 7 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 \\ 0 & 51 & 34 \\ 7 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & \frac{2}{3} \\ 7 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & \frac{2}{3} \\ 0 & -33 & -22 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

whose null vectors have the form

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{t}{3} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$$

Specifying $t = 3$ gives the nontrivial linear combination $\mathbf{v}_1 - 2\mathbf{v}_2 + 3\mathbf{v}_3 = \mathbf{0}$ as before.

3. *(20 points)* In each case a linear system is given, along with its augmented matrix of coefficients, and the resulting reduced row echelon form. Write down the general solution in each case.

(a)
$$\left.\begin{aligned} 3x - 3y + 4z &= 24 \\ 2x - 2y + 3z &= 17 \end{aligned}\right\}$$

$$\left[\begin{array}{ccc|c} 3 & -3 & 4 & 24 \\ 2 & -2 & 3 & 17 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Introducing the parameter t for the free variable y, we obtain

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t+4 \\ t \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

(b)
$$\left.\begin{aligned} x + 3y + 5z - 14w &= 19 \\ -x + 2y + 5z - 6w &= 8 \\ 2x - y - 4z &= 0 \end{aligned}\right\}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & -14 & 19 \\ -1 & 2 & 5 & -6 & 8 \\ 2 & -1 & -4 & 0 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

This system is inconsistent. Since the last equation to be solved is $0 = 1$, there are no solutions.

4. *(25 points)* Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 3 & 3 & 3 \\ 7 & 7 & 0 & 0 & 0 \\ 7 & 7 & 0 & 0 & 0 \\ 0 & 0 & 8 & 8 & 8 \\ 0 & 0 & 8 & 8 & 8 \end{bmatrix}.$$

(a) What is the rank of A?

A has rank 2 (see (c), (d)).

(b) What is the dimension of the null space of A?

By (a), Nul A has dimension $5 - 2 = 3$.

(c) Write down a basis for the row space of A.

By inspection, $(0, 0, 1, 1, 1)$, $(1, 1, 0, 0, 0)$ form a basis for Row A.

(d) Write down a basis for the column space of A.

By inspection, $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 8 \\ 8 \end{bmatrix}$ form a basis for Col A.

(e) Write down a basis for the null space of A.

By inspection, $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ form a basis for Nul A.

5. Answer TRUE or FALSE to each of the following statements.

(a) T (b) T (c) F (d) T (e) T (f) T (g) T (h) F (i) T (j) F (k) T

(a) If $\mathbf{u}, \mathbf{v}$ are two linearly independent vectors in $\mathbb{R}^3$, then there must exist a vector $\mathbf{w}$ in $\mathbb{R}^3$ for which $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is a basis of $\mathbb{R}^3$.

Let $\mathbf{w} \in \mathbb{R}^3$ be any vector outside the plane Span$\{\mathbf{u}, \mathbf{v}\}$.

(b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ are four vectors in $\mathbb{R}^3$, then these four vectors must be linearly dependent.

The linear system $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} + d\mathbf{x} = \mathbf{0}$ has a nonzero solution for (a, b, c, d). Here the coefficient matrix is 3×4, so its reduced row echelon form has at most three pivots and therefore at least one free variable, meaning there are infinitely many solutions.

(c) If $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ are four vectors in $\mathbb{R}^3$, then these four vectors must span $\mathbb{R}^3$.

We can easily choose all four vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ in the (x, y)-plane (or any plane through the origin), so their span is contained in this plane.

(d) If $T : \mathbb{R}^5 \to \mathbb{R}^7$ is a linear transformation, where $\mathbb{R}^5$ consists of 5×1 column vectors and $\mathbb{R}^7$ consists of 7×1 column vectors, then there exists a 7×5 matrix A such that $T\mathbf{x} = A\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^5$.

As explained in class, A is the matrix whose columns are the images $T\mathbf{e}_i$ of the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_5$ of $\mathbb{R}^5$.

(e) If A is an $m \times n$ matrix and $\mathbf{v}$ is a column vector in $\mathbb{R}^n$, then the vector $A\mathbf{v} \in \mathbb{R}^m$ is a linear combination of the columns of A.

As explained in class, denoting the n columns of A by $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and the n entries of $\mathbf{v}$ by $a_1, \ldots, a_n$, we have $A\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$.

(f) If $\mathbf{x}$ is in $\mathrm{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, then we must have $\mathrm{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\} = \mathrm{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

We are given that $\mathbf{x} = a\mathbf{u}+b\mathbf{v}+c\mathbf{w}$ for some $a, b, c \in \mathbb{R}$. Clearly

$$\mathrm{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \subseteq \mathrm{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\}$$

since for all choices of scalars $c_1, \ldots, c_4$, we have

$$c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} + 0\mathbf{x}.$$

Conversely,

$$\mathrm{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\} \subseteq \mathrm{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$$

since for all choices of scalars $c_1, \ldots, c_4$, we have

$$\begin{aligned} c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} + c_4\mathbf{x} &= c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} + c_4(a\mathbf{u}+b\mathbf{v}+c\mathbf{w}) \\ &= (c_1+c_4a)\mathbf{u} + (c_2+c_4b)\mathbf{v} + (c_3+c_4c)\mathbf{w}\,. \end{aligned}$$

(g) If M is an $m \times n$ matrix and A is its reduced row-echelon form, then every row of M must be a linear combination of the rows of A.

As explained in class. At each step during row reduction, the span of the rows is unchanged.

(h) If M is an $m \times n$ matrix and A is its reduced row-echelon form, then every column of M must be a linear combination of the columns of A.

For example, $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] \sim \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Here the original matrix has $\mathrm{Span}\left\{\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]\right\}$ (the line $y = x$) as the span as its columns; but the reduced row echelon form has $\mathrm{Span}\left\{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right\}$ (the x-axis) as the span as its columns.

(i) The rows of the matrix $\left[\begin{smallmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{smallmatrix}\right]$ are linearly independent.

Clearly neither row is a scalar multiple of the other.

(j) The columns of the matrix $\left[\begin{smallmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{smallmatrix}\right]$ are linearly independent.

The third column is the zero vector, which is a trivial linear combination of the first two columns.

(k) The set of vectors of the form $(3s+5t^3,\ 2s-t^3,\ 7s)$ (where $s, t \in \mathbb{R}$ are arbitrary) is a subspace of $\mathbb{R}^3$.

The vectors $s\mathbf{u} + t^3\mathbf{v}$ (where $\mathbf{u} = (3, 2, 7)$ and $\mathbf{v} = (5, -1, 0)$) consist of all vectors in $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$, a plane through the origin in $\mathbb{R}^3$, i.e. a 2-dimensional subspace.