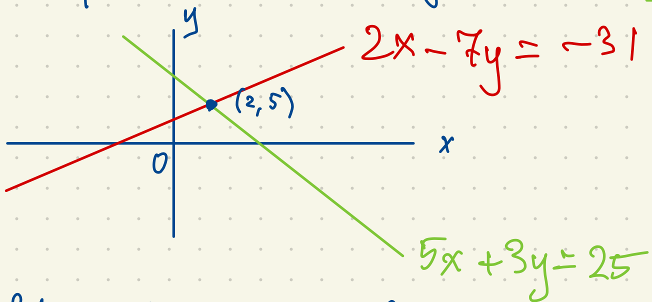
Linear Algebra

Book 1

Example: Find all (x,y) such that $5x+3y=25$ and $2x-7y=-31$.



We are asking for the simultaneous solution of a system of two equations in two unknowns x and y.

$$\begin{cases} 5x+3y=25 & (1) \\ 2x-7y=-31 & (2) \end{cases}$$

$2\times(1) - 5\times(2) = (3)$

$2\times 3 - 5(-7) = 6+35 = 41$

$2\times 25 - 5\times(-31) = 50+155 = 205$

$41y = 205$

$(4) = (3) \div 41$

$y = 5$

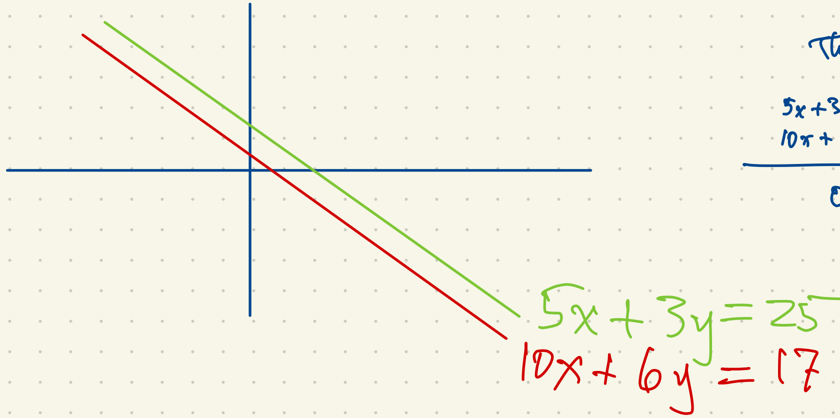
$5x + 15 = 25$

$5x = 10$

$x = 2$

Solution: $(x,y) = (2,5)$ is the unique solution.

Example: Find all (x,y) such that $5x+3y=25$ and $10x+6y=17$.



This system is inconsistent: it has no solution.

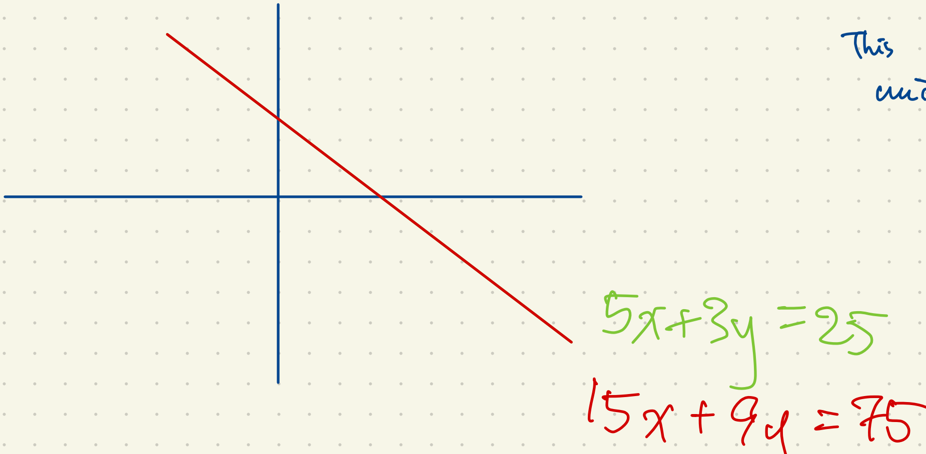
$5x+3y=25$ (1)

$10x+6y=17$ (2)

$0 = 33$ $\quad 2\times(1) - (2)$

This is inconsistent.

Example: Find all (x,y) such that $5x+3y=25$ and $15x+9y=75$.



This system is consistent but the solution is not unique: there are infinitely many solutions.

$$\begin{aligned} 5x+3y&=25 \quad (1)\\ 15x+9y&=75 \quad (2)\\ \hline 0&=0 \quad (3) = 3\times(1)-(2)\end{aligned}$$

A system of m linear equations in n unknowns has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1\\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2\\ \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

(a_{ij}, b_i constants for $i \in \{1,\dots,m\}$, $j = \{1,2,\dots,n\}$; $x_1,\dots,x_n$ variables representing unknowns).

Typically, when $m=n$ we can expect a unique solution;
$m>n$ ... : no solution (inconsistent system);
$m<n$ ... : more than one solution.

Example with $m=n=3$: a system of 3 linear equations in 3 unknowns.

Kim buys a bag of 26 items weighing 226 oz. costing \$34. The items included

cans of tuna (\$1 each, 5 oz each)
apples (\$1 each, 8 oz each)
loaves of bread (\$3 each, 20 oz each)

How many of each item did Kim buy? (say x cans of tuna, y apples, z loaves of bread)

$$\begin{aligned} x + y + z &= 26 && (1)\\ 5x + 8y + 20z &= 226 && (2)\\ x + y + 3z &= 34 && (3) \end{aligned}$$

$$\begin{aligned} 2z &= 8 && (3)-(1) = (4)\\ z &= 4 && (5)\\ x + y &= 22 && (6) = (8)-(5)\\ 5x + 8y &= 146 && (7)\\ 3y &= 36 && (7) - 5\times(6) = (8)\\ y &= 12 && (9) = (8)\div 3\\ x &= 10 && (10) = (6)-(9). \end{aligned}$$

$146 - 5\times 22 = 146 - 110 = 36$

The unique solution of this system is $(x,y,z) = (10,12,4)$. (Kim bought 10 cans of tuna, 12 apples, and 4 loaves of bread.)

Check: that all three equations are satisfied.

Matrix formulation of linear systems

$$\begin{aligned} x + y + z &= 26 \\ 5x + 8y + 20z &= 226 \\ x + y + 3z &= 34 \end{aligned} \longrightarrow \begin{array}{cccc} x & y & z & \text{total} \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 1 & 1 & 3 & 34 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 1 & 1 & 3 & 34 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 0 & 0 & 2 & 8 \end{array}\right]$$

subtract row 1 from row 3