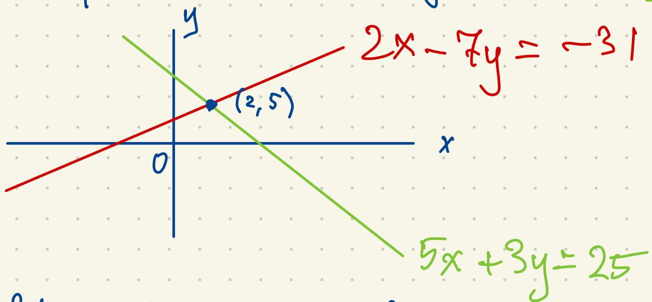
Linear Algebra

Book 1

Example: Find all (x,y) such that $5x+3y=25$ and $2x-7y=-31$.



We are asking for the simultaneous solution of a system of two equations in two unknowns x and y.

$$\begin{cases} 5x+3y=25 & (1) \\ 2x-7y=-31 & (2) \end{cases}$$

$2\times(1) - 5\times(2) = (3)$

$(4) = (3) \div 41$

$2\times 3 - 5(-7) = 6+35 = 41$

$2\times 25 - 5\times(-31) = 50+155 = 205$

$41y = 205$

$y = 5$

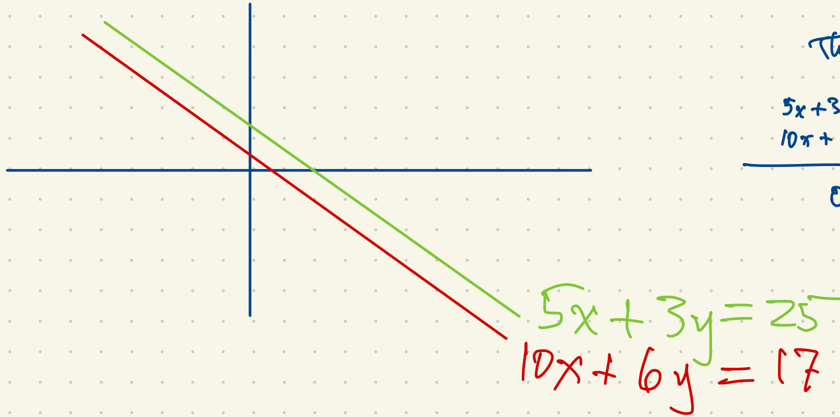
$5x + 15 = 25$

$5x = 10$

$x = 2$

Solution: $(x,y) = (2,5)$ is the unique solution.

Example: Find all (x,y) such that $5x+3y=25$ and $10x+6y=17$.



This system is inconsistent: it has no solution.

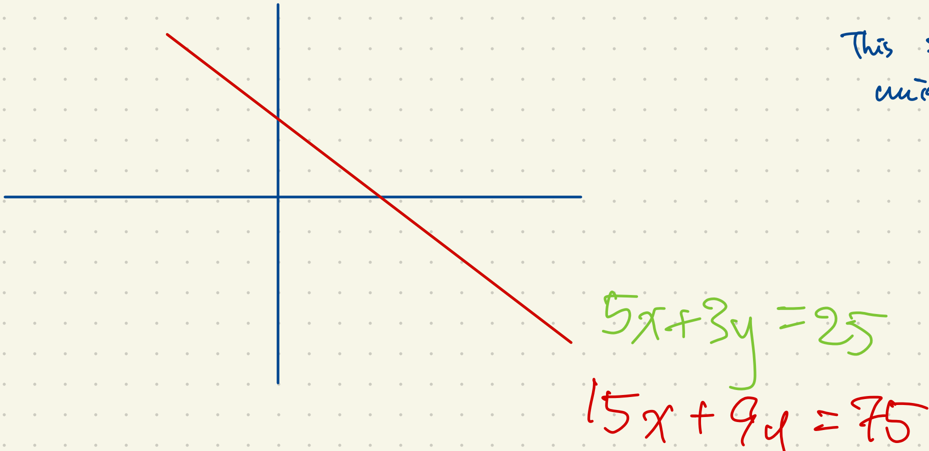
$5x+3y=25$ (1)

$10x+6y=17$ (2)

$0 = 33$ $\quad 2\times(1) - (2)$

This is inconsistent.

Example : Find all (x,y) such that $5x+3y=25$ and $15x+9y=75.$



This system is consistent but the solution is not unique: there are infinitely many solutions.

$$\begin{array}{rl} 5x+3y=25 & (1) \\ 15x+9y=75 & (2) \\ \hline 0=0 & (3) = 3\times(1)-(2) \end{array}$$

A system of m linear equations in n unknowns has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

(a_{ij}, b_i constants for $i \in \{1,\cdots,m\}$, $j = \{1,2,\cdots,n\}$; $x_1,\cdots,x_n$ variables representing unknowns).

Typically, when $m=n$ we can expect a unique solution;
$m>n$ " — — — " no solution (inconsistent system);
$m<n$ " — — " more than one solution.

Example with $m=n=3$: a system of 3 linear equations in 3 unknowns.

Kim buys a bag of 26 items weighing 226 oz. costing \$34. The items included

cans of tuna (\$1 each, 5oz each)
apples (\$1 each, 8oz each)
loaves of bread (\$3 each, 20oz each)

How many of each item did Kim buy? (say x cans of tuna, y apples, z loaves of bread)

$$
\begin{aligned}
x + y + z &= 26 && (1)\\
5x + 8y + 20z &= 226 && (2)\\
x + y + 3z &= 34 && (3)
\end{aligned}
$$

$$
\begin{aligned}
2z &= 8 && (3)-(1) = (4)\\
z &= 4 && (5)\\
x + y &= 22 && (6) = (8)-(5)\\
5x + 8y &= 146 && (7)\\
3y &= 36 && (7) - 5\times(6) = (8)\\
y &= 12 && (9) = (8)\div 3\\
x &= 10 && (10) = (6)-(9).
\end{aligned}
$$

$146 - 5\times 22 = 146 - 110 = 36$

The unique solution of this system is $(x,y,z) = (10,12,4)$. (Kim bought 10 cans of tuna, 12 apples, and 4 loaves of bread.)

Check: that all three equations are satisfied.

Matrix formulation of linear systems

$$\begin{aligned} x + y + z &= 26 \\ 5x + 8y + 20z &= 226 \\ x + y + 3z &= 34 \end{aligned} \longrightarrow \begin{array}{cccc} x & y & z & \text{total} \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 1 & 1 & 3 & 34 \end{array}\right]$$

$$226 - 5 \times 26 = 226 - 130 = 96$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 1 & 1 & 3 & 34 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 0 & 0 & 2 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 0 & 3 & 15 & 96 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 0 & 1 & 5 & 32 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

subtract row 1 from row 3 — divide row 3 by 2 — subtract 5 times row 1 from row 2 — divide row 2 by 3

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 14 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{array}\right] \quad \text{i.e.} \quad \begin{aligned} x &= 10 \\ y &= 12 \\ z &= 4 \end{aligned}$$

subtract 5 times row 3 from row 2 — subtract row 2 from row 1 — subtract row 3 from row 1

Example: Find all (x, y) such that $5x + 3y = 25$ and $2x - 7y = -31$.

$$-7 - \frac{6}{5} = -\frac{35}{5} - \frac{6}{5} = -\frac{41}{5}$$

$$-31 - 10 = -41$$

$$\begin{array}{cc} x & y \end{array} \left[\begin{array}{cc|c} 5 & 3 & 25 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3/5 & 5 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{3}{5} & 5 \\ 0 & -\frac{41}{5} & -41 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{3}{5} & 5 \\ 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$$

divide row 1 by 5 — subtract 2 times row 1 from row 2 — multiply row 2 by $-\frac{5}{41}$ — subtract $\frac{3}{5}$ times row 2 from row 1

Solution: $(x, y) = (2, 5)$.

Alternatively:

$$\left[\begin{array}{cc|c} 5 & 3 & 25 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -7 & -31 \\ 5 & 3 & 25 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{7}{2} & -\frac{31}{2} \\ 5 & 3 & 25 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$$

interchange rows 1 and 2

Even better: $\left[\begin{array}{cc|c} 5 & 3 & 25 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 17 & 87 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 17 & 87 \\ 0 & -41 & -205 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 17 & 87 \\ 0 & 1 & 5 \end{array}\right]$

subtract 2 times row 2 from row 1 — subtract 2 times row 1 from row 2 — divide row 2 by -41

$-31 - 2\times 87$
$= -31 - 174$
$= -205$

$\sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$

subtract 17 times row 2 from row 1

Solution: $(x,y) = (2,5)$.

Check! $5\times 2 + 3\times 5 = 25$
$2\times 2 - 7\times 5 = -31$

Elementary row operations:
- (i) add a multiple of one row to another
- (ii) multiply a row by a nonzero constant
- (iii) interchange two rows

$A \sim B$ means that A, B are linear systems having the same solutions.

We use Gaussian elimination to reduce $A_1 \sim A_2 \sim \cdots \sim A_m$ where A_1 represents the linear system and A_m represents an equivalent linear system (ie. having the same solutions) but A_m is simpler than A_1. Each step $A_i \sim A_{i+1}$ is obtained by one elementary row operation.

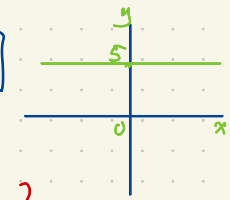
why just one operation at a time?

$$\left.\begin{array}{l} 5x + 3y = 25 \\ 2x - 7y = -31 \end{array}\right\} \quad \underbrace{\left[\begin{array}{cc|c} 5 & 3 & 25 \\ 2 & -7 & -31 \end{array}\right]} \sim \left[\begin{array}{cc|c} 1 & \frac{3}{5} & 5 \\ 1 & -\frac{7}{2} & -\frac{31}{2} \end{array}\right] \not\sim \left[\begin{array}{cc|c} 0 & \frac{41}{10} & \frac{41}{2} \\ 0 & -\frac{41}{10} & -\frac{41}{2} \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 5 \\ 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

divide row 1 by 5
divide row 2 by 2

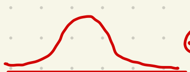
subtract row 2 from row 1
subtract row 1 from row 2

ie. $\underbrace{\begin{array}{l} y = 5 \\ 0 = 0 \end{array}}$



unique solution $(2,5)$

infinitely many solutions

Gauss  Gaussian distribution

$\left[\begin{array}{cc|c} 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$, $\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 0 & 4 \end{array}\right]$, $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$ are examples of matrices in reduced row echelon form:

they cannot be simplified any further by elementary row operations.

$\left[\begin{array}{cc|c} 1 & 17 & 87 \\ 0 & 1 & 5 \end{array}\right]$ is almost reduced; it is in row echelon form.

For a linear system whose matrix is in row echelon form, we can solve for the unknowns $x_1, x_2, \cdots, x_n$; we solve for x_n, then x_{n-1}, then x_{n-2}, $\cdots$, x_1 by back-substitution.

eg.
$$\begin{array}{c} \begin{array}{ccc|c} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 5 & 3 & 7 & 3 \\ 0 & 2 & 11 & 4 \\ 0 & 0 & 6 & 8 \end{array}\right] \end{array}$$
is in row echelon form.

Every linear system has a unique reduced row echelon form.

In any $m \times n$ matrix, a pivot is the first nonzero entry in its row.
(Pivots are highlighted above.)

In order for a matrix to be in row echelon form, we must have
- pivots in any row must occur to the right of pivots in any previous rows;
- any zero rows occur at the bottom.

Assuming a matrix is already in row echelon form, then to be in reduced row echelon form, we must have
- every pivot entry must be a 1
- every column having a pivot has only one nonzero entry.

Example: Solve the following linear system of 3 equations in 5 unknowns:

$$\begin{cases} x_1 + 4x_2 - x_3 + 2x_4 + 3x_5 = 6 \\ 2x_1 + 8x_2 - x_3 + 7x_4 + 4x_5 = 19 \\ -x_1 - 4x_4 + 4x_3 + 8x_4 - 4x_5 = 26 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 2 & 8 & -1 & 7 & 4 & 19 \\ -1 & -4 & 4 & 8 & -4 & 26 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ -1 & -4 & 4 & 8 & -4 & 26 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 3 & 10 & -1 & 32 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right]$$

This matrix is in row echelon form. This can be used to solve the linear system by back-substitution.

$x_4 + 5x_5 = 11$ $\qquad x_5 = t$ is a free parameter.

$x_4 = 11 - 5t$

$x_3 + 3x_4 - 2x_5 = 7$

$x_3 = 7 - 3x_4 + 2x_5 = 7 - 3(11-5t) + 2t = -26 + 17t$

$x_1 + 4x_2 - x_3 + 2x_4 + 3x_5 = 6$

$x_2 = s$ is another free parameter

$x_1 = 6 - 4x_2 + x_3 - 2x_4 - 3x_5 = 6 - 4s + (-26+17t) - 2(11-5t) - 3t$

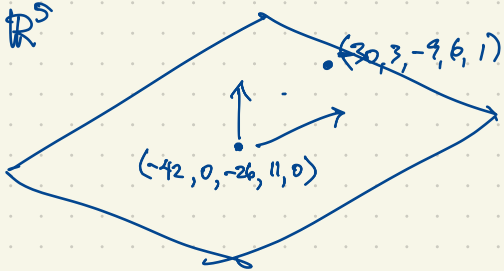
$= -42 - 4s + 24t$

Solutions: $(x_1, x_2, x_3, x_4, x_5) = (-42 - 4s + 24t,\ s,\ -26 + 17t,\ 11 - 5t,\ t)$ where s,t are arbitrary.

Geometrically, the set of solutions forms a plane (2-dimensional surface) in $\mathbb{R}^5$.

two parameters s,t are coordinates for the plane

$\mathbb{R}^5$



Solution set inside $\mathbb{R}^5$.

The (solution) point corresponding to $(s,t) = (3,1)$ is $(30, 3, -9, 6, 1)$ is another solution.

Our system is consistent but the solution is not unique.

$$\left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 5 & 1 & 13 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 0 & -24 & -42 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 0 & -24 & -42 \\ 0 & 0 & 1 & 0 & -17 & -26 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right]$$

(row echelon form) (reduced row echelon form)

To solve a linear system in reduced row echelon form, introduce parameters for the free variables (the variables whose columns do not contain a pivot).

In the example above, x_2 and x_5 are the free variables. Introduce s, t. $x_2 = s$, $x_5 = t$ can be chosen freely. Solve for the variables x_1, x_3, x_4 using the equations appearing in the reduced row echelon form:

$$\left.\begin{aligned} x_1 + 4s - 24t &= -42 \\ x_3 \quad -17t &= -26 \\ x_4 + 5t &= 11 \end{aligned}\right\} \Rightarrow (x_1, x_2, x_3, x_4, x_5) = (-42 - 4s + 24t,\ s,\ -26 + 17t,\ 11 - 5t,\ t)$$

where s, t are arbitrary.

(This is the parametric solution in terms of the parameters s, t. The system is consistent, having infinitely many solutions.)

As long as the rightmost column has no pivot, the system is consistent.

The general solution can be written as

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5) &= (-42 - 4s + 24t,\ s,\ -26 + 17t,\ 11 - 5t,\ t) \\ &= (-42, 0, -26, 11, 0) + s(-4, 1, 0, 0, 0) + t(24, 0, 17, -5, 1)\end{aligned}$$

$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$ (vector addition)

$\underset{\text{scalar}}{c}\,\underbrace{(a_1, a_2, \dots, a_n)}_{\text{vector}} = (ca_1, ca_2, \dots, ca_n)$ (scalar multiplication)

Algebraic operations for matrices

If $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix}$ then $BA = \underbrace{\begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \end{bmatrix}}_{2\times 3} = \underbrace{\begin{bmatrix} 13 & 11 & 41 \\ -1 & -27 & 23 \end{bmatrix}}_{2\times 3}$

Here AB is undefined.

An $m\times n$ matrix has the form

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

$a_{i,j}$ is the (i,j)-entry of the matrix A.

$i \in \{1, 2, \ldots, m\}$

$j \in \{1, 2, \ldots, n\}$.

Often $a_{i,j}$ is written a_{ij}. (unless this results in confusing)

If A is $m\times n$ and B is $n\times r$ then $\underbrace{A}_{m\times n}\underbrace{B}_{n\times r}$ is $m\times r$.

We can't multiply two matrices unless the number of columns in the first matrix equals the number of rows in the second matrix.

eg. $\underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}}_{3\times 2} = \begin{bmatrix} 12 & 8 \\ 23 & 4 \end{bmatrix}$ whereas $\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \end{bmatrix}}_{2\times 3} = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \\ 5 & -1 & 21 \end{bmatrix}$

If $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$ then $A^2 = \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}}_{A} = \begin{bmatrix} 7 & 3 \\ 1 & 4 \end{bmatrix}$, $A^3 = A^2 A = \begin{bmatrix} 7 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 17 & 18 \\ 6 & -1 \end{bmatrix}$

Recall: the linear system $\left.\begin{matrix} 5x+3y=25 \\ 2x-7y=-31 \end{matrix}\right\}$ has a unique solution $(x,y)=(2,5)$.

One way to solve this: Write the linear system as $Av=b$ where A is a 2×2 matrix, $v=\begin{bmatrix} x \\ y \end{bmatrix}$ is a 2×1 matrix (i.e. column vector of length 2) and $b=\begin{bmatrix} 25 \\ -31 \end{bmatrix}$ is a 2×1 matrix of constants.

Here $A=\begin{bmatrix} 5 & 3 \\ 2 & -7 \end{bmatrix}$.

$Av=b$ says $\underbrace{\begin{bmatrix} 5 & 3 \\ 2 & -7 \end{bmatrix}}_{2\times \cancel{7}}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\cancel{2}\times 1}=\underbrace{\begin{bmatrix} 25 \\ -31 \end{bmatrix}}_{2\times 1}$ i.e. $\begin{bmatrix} 5x+3y \\ 2x-7y \end{bmatrix}=\begin{bmatrix} 25 \\ -31 \end{bmatrix}$

Compare: To solve $3x=5$, multiply both sides by $3^{-1}=\frac{1}{3}$ on the left: $\cancel{3^{-1}}\cancel{3}x=3^{-1}5$ i.e. $x=\frac{5}{3}$.

To solve $Av=b$, multiply both sides on the left by $A^{-1}=\frac{1}{41}\begin{bmatrix} 7 & 3 \\ 2 & -5 \end{bmatrix}=\begin{bmatrix} \frac{7}{41} & \frac{3}{41} \\ \frac{2}{41} & -\frac{5}{41} \end{bmatrix}$

$Av=b$

$A^{-1}Av=A^{-1}b$

$$\frac{1}{41}\begin{bmatrix} 7 & 3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 2 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\frac{1}{41}\begin{bmatrix} 7 & 3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} 25 \\ -31 \end{bmatrix}$$

$$\frac{1}{41}\begin{bmatrix} 41 & 0 \\ 0 & 41 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\frac{1}{41}\begin{bmatrix} 82 \\ 205 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}=\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\substack{I_2 \text{ is the} \\ 2\times 2 \text{ identity matrix}}}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$I_n=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$ ($n\times n$ identity matrix)

If $\underbrace{\underbrace{(A B)}_{2\times 3} C}_{2\times 5}$ with A: 2×7, B: 7×3, C: 3×5, $= \underbrace{A\,(BC)}_{2\times 5}$ with A: 2×7, BC: 7×5, by associativity, you can do the first way since that's is faster.

We say A and B commute if $AB = BA$.

Which 2×2 matrices commute with $A = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$? Answer by solving the appropriate linear system of 4 equations in 4 unknowns. Let $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. In order for $AB = BA$ we require

$$\begin{bmatrix} 3x+z & 3y+w \\ 4z & 4w \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3x & x+4y \\ 3z & z+4w \end{bmatrix}$$

i.e. $\begin{cases} 3x+z = 3x \\ 3y+w = x+4y \\ 4z = 3z \\ 4w = z+4w \end{cases}$

$$\begin{array}{cccc|c} x & y & z & w & \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

x, z are basic variables (in the pivot columns)

y, w are free variables (non-pivot columns)

Introduce s, t as parameters. $y = s$, $w = t$

and solve for x, z: $x = -s+t$, $z = 0$ so $B = \begin{bmatrix} -s+t & s \\ 0 & t \end{bmatrix} = s\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(a <u>linear combination</u> of $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

Check: If $s=1, t=0$ then $B = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$ and $BA = AB$.

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}$$

$$IA = A = AI$$

For Friday's Quiz:
- Representation of linear systems in matrix form. $Ax = b$
- Particular and general solutions
- Null vectors
- Homogeneous linear systems $Ax = 0$

homogenize

$$\underset{\substack{\uparrow \\ m\times n}}{\mathrm{Nul}(A)} = \{\underbrace{x}_{n\times 1} \in \mathbb{R}^n : \underbrace{A}_{m\times n}\underbrace{x}_{n\times 1} = \underbrace{0}_{m\times 1} \in \mathbb{R}^m\}$$

is the <u>null space</u> of A. Its vectors are called null vectors.

$$\begin{cases} x_1 + 4x_2 - x_3 + 2x_4 + 3x_5 = 6 \\ 2x_1 + 8x_2 - x_3 + 7x_4 + 4x_5 = 19 \\ -x_1 - 4x_4 + 4x_3 + 8x_4 - 4x_5 = 26 \end{cases}$$

$$\underbrace{\begin{bmatrix} 1 & 4 & -1 & 2 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}}_{3\times 5} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{5\times 1} = \underbrace{\begin{bmatrix} 6 \\ 7 \\ 11 \end{bmatrix}}_{3\times 1}$$

Every linear system has the form $\underbrace{A}_{m\times n}\underbrace{x}_{n\times 1} = \underbrace{b}_{m\times 1}$ (for m linear equations in n unknowns).

$$A = \begin{bmatrix} 1 & 4 & -1 & 2 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 7 \\ 11 \end{bmatrix}$$

Some prefer to write $\vec{x}$ or $\overset{\rightharpoonup}{x}$ instead of x. or even $\mathbf{x}$ (bold face) or $\underset{\sim}{x}$. For us, context is used to determine whether we are talking about a matrix, a vector, a scalar, a set, a linear transformation, a vector space, etc.

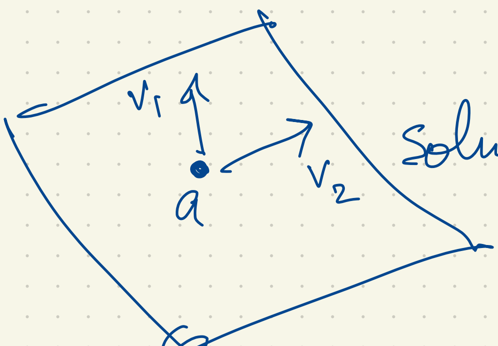
Some linear systems are inconsistent (meaning that they have no solutions). If a linear system $Ax=b$ is consistent then its solutions have the form $x = a + c_1v_1 + c_2v_2 + \cdots + c_kv_k$ for some particular solution $x=a$; $c_1, \ldots, c_k$ scalars (constants) and $v_1, \ldots, v_k$ are independent solutions of $Ax=0$.

The solutions of the example $Ax=b$ above have the form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \boxed{\underbrace{\begin{bmatrix} -42 \\ 0 \\ -26 \\ 11 \\ 0 \end{bmatrix}}_{a} + s\underbrace{\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + t\underbrace{\begin{bmatrix} 24 \\ 0 \\ 17 \\ -5 \\ 1 \end{bmatrix}}_{v_2} = \underbrace{\begin{bmatrix} -42 - 4s + 24t \\ s \\ -26 + 17t \\ 11 - 5t \\ t \end{bmatrix}}_{5\times 1}}$$

a particular solution

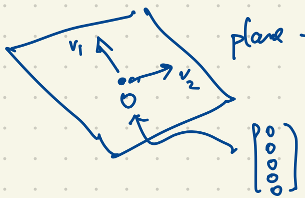
The general solution



solution set (i.e. the set of all solutions)

$A = \begin{bmatrix} 1 & 4 & -1 & 2 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$ gives rise to a homogeneous linear system $A x = \underbrace{0}_{3\times 1}$ i.e. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e. $\left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \end{array}\right]$ has solutions $x = s v_1 + t v_2$ (as $a = \begin{bmatrix} -42 \\ \vdots \\ \vdots \end{bmatrix}$ here)



plane through the origin in $\mathbb{R}^5$
(solutions of $Ax = 0$)

Systems of the form $Ax = 0$ are called <u>homogeneous</u> meaning if u and v are solutions then so is $su + tv$.
(i.e. $Au = 0$ and $Av = 0$ then $A(su + tv) = 0$
$sAu + tAv = 0$

A homogeneous system $Ax = 0$ is always consistent, since the zero vector $x = 0$ is a solution.

$\mathrm{Nul}(A)$ = null space of A = $\{$ all solutions of the homogeneous system $\underbrace{A}_{m\times n}\underbrace{x}_{n\times 1} = \underbrace{0}_{m\times 1} \} \subseteq \mathbb{R}^n$

$x \in \mathbb{R}^n$ $\quad 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^m$

$\mathrm{Nul}(A)$ might be: the origin in $\mathbb{R}^n$,
or maybe a line through the origin in $\mathbb{R}^n$,
... - - - . plane . - - . - -

Checking answers: If we reduce $A \sim A'$, how can we check our work? (A' could be a row echelon form for A, or maybe a reduced row echelon form for A). A and A' have the same null vectors.

eg. $A = \begin{bmatrix} 1 & 4 & -1 & 2 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \sim A' = \begin{bmatrix} 1 & 4 & 0 & 0 & -24 \\ 0 & 0 & 1 & 0 & -17 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$

A' has null vectors $\begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 24 \\ 0 \\ 17 \\ -5 \\ 1 \end{bmatrix}$. Check that they are also null vectors for A.

$$\begin{bmatrix} 1 & 4 & 0 & 0 & -24 \\ 0 & 0 & 1 & 0 & -17 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 24 \\ 0 \\ 17 \\ -5 \\ 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Every null vector for B is also a null vector for AB.

If $Bx = 0$ then $ABx = A0 = 0$

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} -1/3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 5 \\ 0 & 1 & 2 \\ 9 & 8 & 19 \end{bmatrix} \begin{bmatrix} -1/3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Last week's quiz: $\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix}}_{B} = \begin{bmatrix} 3 & 2 & 5 \\ 0 & 1 & 2 \\ 9 & 8 & 19 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}}_{A} = \begin{bmatrix} 5 & 6 \\ 12 & 18 \end{bmatrix}$$

If C is an $n \times n$ matrix, the <u>trace</u> of C is the sum of the n entries on the main diagonal of C. (denoted $\operatorname{tr} C$)

If A is $m \times n$ and B is $n \times m$, then $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. = trace

Eg. Solve $y'' + y = x^2$.

A particular solution is $y = x^2 - 2$. Check: $y'' + y = 2 + (x^2 - 2) = x^2$.

The general solution is $y = x^2 - 2 + a\sin x + b\cos x$ where $a, b \in \mathbb{R}$ are arbitrary.

The homogenized equation $y'' + y = 0$ is homogeneous (so if y_1 and y_2 are two solutions, then a linear combination $ay_1 + by_2$ is also a solution). The general solution of $y'' + y = 0$ is $y = a\sin x + b\cos x$.

What about $\sin(10° + x) = (\cos 10°)\sin x + (\sin 10°)\cos x$. (Every solution of $y'' + y = 0$ is a linear combination of $\sin x$ and $\cos x$.)

A linear combination of vectors $v_1, \ldots, v_k$ is a vector of the form $c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$ where $c_1, \ldots, c_k$ are scalars. Moreover, $\sin x$ and $\cos x$ are linearly independent. We cannot express the general solution of $y'' + y = 0$ as a linear combination of fewer than two basic solutions.

It is correct to say that the general solution of $y'' + y = 0$ is a linear combination of $\sin x$, $\cos x$ and $\sin(10° + x)$, i.e. every solution has the form $y = c_1 \sin x + c_2 \cos x + c_3 \sin(10° + x)$. However this is the same as

$\sin(10° + x) = (\cos 10°)\sin x + (\sin 10°)\cos x$

$y = (c_1 + c_3 \cos 10°)\sin x + (c_2 + c_3 \sin 10°)\cos x$.

It's also correct to say: every solution is a linear combination of $\sin x$ and $\sin(10° + x)$.

$$\text{e.g. } \cos x = \underbrace{\left(-\frac{\cos 10°}{\sin 10°}\right)}_{(-\cot 10°)} \sin x + \underbrace{\left(-\frac{1}{\sin 10°}\right)}_{(-\csc 10°)} \sin(10° + x)$$

In fact any two solutions of $y'' + y = 0$ can be used to generate all the other solutions by taking linear combinations, as long as neither of your two particular solutions is a scalar multiple of the other.

A list of vectors $v_1, \ldots, v_k$ is linearly independent if none of them is a linear combination of the others.

Alternatively, the only way to have $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$ is if $c_1 = \cdots = c_k = 0$.

Eg. $\sin x$, $\cos x$, $\sin(10^\circ + x)$ are linearly dependent (not linearly independent) since $\sin(10^\circ + x)$ is a linear combination of $\sin x$ and $\cos x$.

What does it mean for a list of vectors $v_1, \dots, v_k$ to be linearly independent?

For $k=1$: $\{v_1\}$ is linearly independent iff $v_1 \neq 0$. Any list of vectors containing a zero vector is linearly dependent.

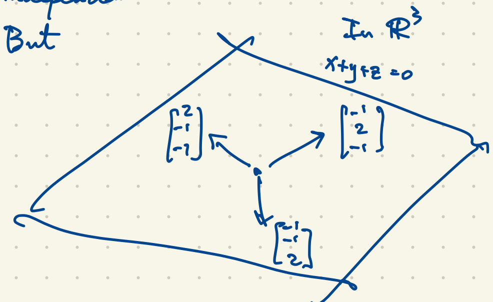
For $k=2$: $\{v_1, v_2\}$ is linearly independent iff $v_1 \neq 0$ and v_2 is not a scalar multiple of v_1.

For $k=3$: $\{v_1, v_2, v_3\}$ is linearly independent iff $v_1 \neq 0$ and v_2 is not a scalar multiple of v_1 and v_3 is not a linear combination of v_1 and v_2.

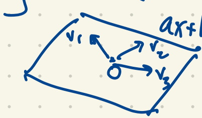
Warning: In $\mathbb{R}^3$, is the set of vectors $\left\{ \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$ linearly independent?

None of these vectors is a scalar multiple of either of the others. But

$$\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{So} \quad \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = -\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$



Any three vectors in the same plane (through the origin) are linearly dependent.



$$(*)v_1 + (*)v_2 + (*)v_3 = 0$$

If $c_1v_1 + c_2v_2 + c_3v_3 = 0$ where c_1, c_2, c_3 are scalars, not all zero then v_1, v_2, v_3 are linearly dependent.

Note that if $c_3 \neq 0$ then $v_3 = -\frac{c_1}{c_3}v_1 - \frac{c_2}{c_3}v_2$ is a linear combination of v_1, v_2.

But if $c_3 = 0$ we have $c_1v_1 + c_2v_2 = 0$. Here if $c_2 \neq 0$ then $v_2 = -\frac{c_1}{c_2}v_1$ is a scalar multiple of v_1.

But if $c_2 = 0$ and $c_3 = 0$ we have $c_1v_1 = 0$. In this case if $c_1 \neq 0$ then $v_1 = 0$ giving a linearly dependent list.

In general, when is a list of vectors $v_1, \dots, v_k$ linearly independent? Iff:

- $v_1 \neq 0$
- v_2 is not a scalar multiple of v_1
- v_3 . . . . . linear combination of v_1, v_2

and so on. (i.e. none of the vectors in the list is a linear combination of the previous vectors in the list.)

For a matrix in row echelon form (in particular reduced row echelon form), all nonzero rows are linearly independent. Also, the pivot columns are linearly independent.

eg. $\begin{bmatrix} 0 & 3 & 1 & 2 & 7 & 11 & 13 & -2 \\ 0 & 0 & 0 & 5 & 6 & 4 & 7 & 6 \\ 0 & 0 & 0 & 0 & 0 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ The first three rows are $(0,3,1,2,7,11,13,-2)$, $(0,0,0,5,6,4,7,6)$, $(0,0,0,0,0,8,4,0)$ are linearly independent.
The four rows are linearly dependent.

The pivot columns are $\begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 11 \\ 4 \\ 8 \\ 0 \end{bmatrix}$ are linearly independent.

If A is any $m \times n$ matrix and $A \sim A'$ then every row of A' is a linear combination of rows of A (and every row of A is a linear combination of rows of A'). The same does not hold for columns

eg. $\begin{bmatrix} 1 & 0 & 3 & 2 \\ 2 & 0 & 6 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

eg. $3\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + 7\begin{bmatrix} 0 & -1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -1 \\ 31 & 10 \end{bmatrix}$

Every polynomial in x is a linear combination of powers of x, eg. $5+3x-7x^3 = 5\cdot x^0 + 3\cdot x^1 + (-7)\cdot x^3$

Let $S = \{u, v, w\}$, $S' = \{u', v', w'\}$. If every vector in S is a linear combination of vectors in S', must every vector in S' be a linear combination of vectors in S? False. For example

$S = \left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} \right\}$, $S' = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

with u, v, w the vectors of S and u', v', w' the vectors of S'.

$u = (0)u' + (-3)v' + (3)w'$
$v = \frac{5}{2}u' + 0v' + 0w'$
$w = \frac{7}{2}u' + 0v' + 0w'$

If x, y are linear combinations of u, v, w, and z is a linear combination of x and y, must z be a linear combination of u, v, w? True. If $x = au + bv + cw$, $y = a'u + b'v + c'w$

$z = rx + sy$
$= r(au + bv + cw) + s(a'u + b'v + c'w)$
$= (ra + sa')u + (rb + sb')v + (rc + sc')w$

Let U, U' be lists of vectors with $U \subseteq U'$. If U is linearly dependent, then U' must be linearly dependent.
True.

A linear transformation T is a function satisfying $T(au+bv) = aT(u) + bT(v)$ for all u, v vectors and a, b scalars.

Eg. Every 2×3 matrix A defines a linear transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^2$ (i.e. $T_A(u) \in \mathbb{R}^2$ whenever $u \in \mathbb{R}^3$).

for $A = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & -5 \end{bmatrix}$, $T_A\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & -5 \end{bmatrix} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\in \mathbb{R}^3} = \underbrace{\begin{bmatrix} u_1 + 3u_2 + 7u_3 \\ 2u_1 \quad - 5u_3 \end{bmatrix}}_{\in \mathbb{R}^2}$.

More generally any $m \times n$ matrix A defines a linear transformation
$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
called a matrix transformation.

$T_A u = Au$ is linear because
$T_A(au + bu') = A(au+bu') = aAu + bAu' = aT_A(u) + bT_A(u')$.

An example of a linear transformation that is not a matrix transformation:
the derivative operator $D = \frac{d}{dx}$

$D(\sin x) = \cos x$

$D(x^3) = 3x^2$

$D(af + bg) = aDf + bDg$

a, b scalars
f, g differentiable functions

$T(f) = f(7)$ is a linear operator
$T(af+bg) = (af+bg)(7) = af(7) + bg(7)$
$= aT(f) + bT(g)$

An example of a nonlinear operator: $T(f) = f(0)f(1)$
$T(3+x) = 3 \cdot 4 = 12 \neq \underbrace{T(3)}_{3\cdot 3 = 9} + \underbrace{T(x)}_{0 \cdot 1 = 0} = 9+0=9$

$T(f) = \int_0^1 f(t)\,dt$ is a linear operator.

$$T(af+bg) = \int_0^1 (af(t)+bg(t))\,dt = a\int_0^1 f(t)\,dt + b\int_0^1 g(t)\,dt = aT(f)+bT(g)$$

$$\left.\begin{array}{l} \sqrt{f+g} \neq \sqrt{f}+\sqrt{g} \\ \dfrac{1}{f+g} \neq \dfrac{1}{f}+\dfrac{1}{g} \end{array}\right\} \text{ nonlinear operations}$$