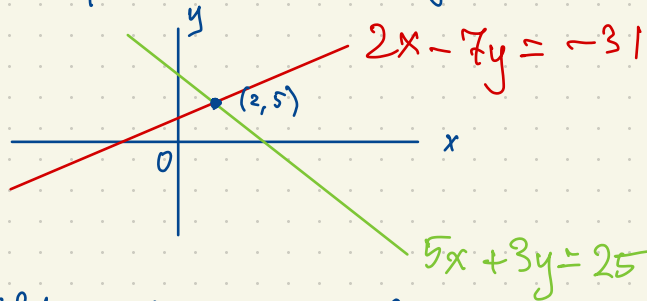


The background of the entire image is a repeating pattern of triangles. The triangles are arranged in a grid, with each triangle pointing upwards. The color of the triangles is a muted, earthy brown or taupe, set against a light beige or off-white background. The pattern is consistent across the entire image.

Linear Algebra

Book 1

Example: Find all (x, y) such that $\underline{5x+3y=25}$ and $\underline{2x-7y=-31}$.



We are asking for the simultaneous solution of a system of two equations in two unknowns x and y .

$$\begin{cases} 5x + 3y = 25 & (1) \\ 2x - 7y = -31 & (2) \end{cases}$$

$$\begin{aligned} 11y &= 205 \\ y &= 5 \end{aligned}$$

$$\begin{aligned} 5x + 15 &= 25 \\ 5x &= 10 \\ x &= 2 \end{aligned}$$

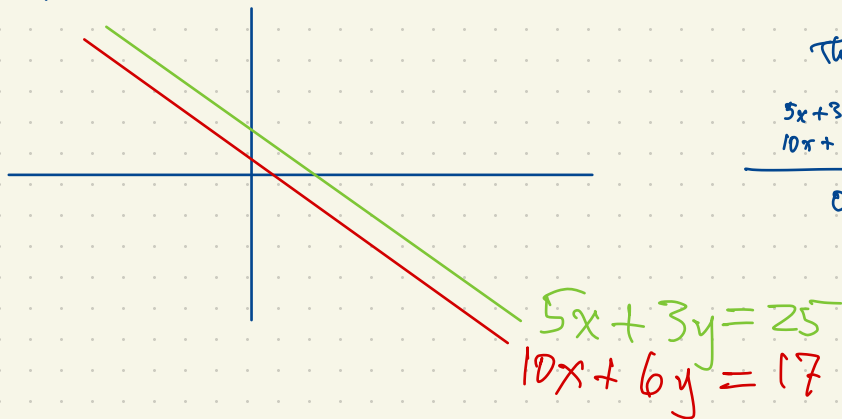
$$\begin{aligned} 2 \times (1) - 5 \times (2) &= (3) \\ (1) &= (3) \div 41 \end{aligned}$$

$$2 \times 3 - 5(-7) = 6 + 35 = 41$$

$$2 \times 25 - 5 \times (-31) = 50 + 155 = 205$$

Solution: $(x, y) = (2, 5)$ is the unique solution.

Example: Find all (x, y) such that $\underline{5x+3y=25}$ and $\underline{10x+6y=17}$.



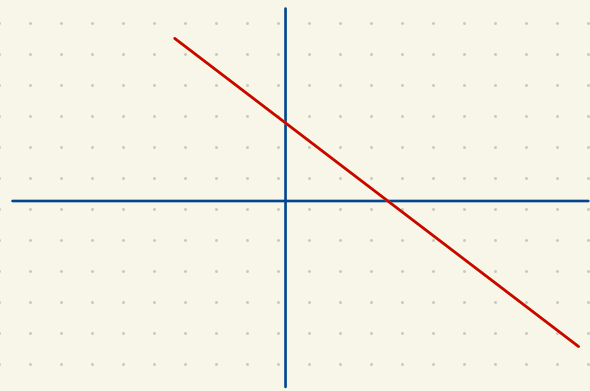
This system is inconsistent: it has no solution.

$$\begin{cases} 5x + 3y = 25 & (1) \\ 10x + 6y = 17 & (2) \end{cases}$$

$$0 = 33 \quad 2 \times (1) - (2)$$

This is inconsistent.

Example: Find all (x, y) such that $5x + 3y = 25$ and $15x + 9y = 75$.



This system is consistent but the solution is not unique: there are infinitely many solutions.

$$\begin{array}{rcl} 5x + 3y = 25 & (1) \\ 15x + 9y = 75 & (2) \\ \hline 0 = 0 & (3) = 3 \times (1) - (2) \end{array}$$

$$5x + 3y = 25$$
$$15x + 9y = 75$$

A system of m linear equations in n unknowns has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$(a_{ij}, b_i \text{ constants for } i \in \{1, \dots, m\}, j \in \{1, 2, \dots, n\}; x_1, \dots, x_n \text{ variables representing unknowns}).$

Typically, when $m = n$ we can expect a unique solution;
 $m > n$: - - - no solution (inconsistent system);
 $m < n$: - - - more than one solution.

Example with $m=n=3$: a system of 3 linear equations in 3 unknowns.
Kim buys a bag of 26 items weighing 226 oz. costing \$34. The items included

cans of tuna (\$1 each, 5oz each)

apples (\$1 each, 8oz each)

loaves of bread (\$3 each, 20oz each)

How many of each item did Kim buy? (say x cans of tuna, y apples, z loaves of bread)

$$x + y + z = 26 \quad (1)$$

$$5x + 8y + 20z = 226 \quad (2)$$

$$x + y + 3z = 34 \quad (3)$$

$$2z = 8 \quad (3) - (1) = (4)$$

$$z = 4 \quad (5)$$

$$x + y = 22 \quad (6) = (8) - (5)$$

$$5x + 8y = 146 \quad (7)$$

$$3y = 36 \quad (7) - 5 \times (6) = (8)$$

$$y = 12 \quad (9) = (8) \div 3$$

$$x = 10 \quad (10) = (6) - (9)$$

$$146 - 5 \times 22 = 146 - 110 = 36$$

The unique solution of this system is $(x, y, z) = (10, 12, 4)$.

(Kim bought 10 cans of tuna, 12 apples, and 4 loaves of bread.)

Check! that all three equations are satisfied.

Matrix formulation of linear systems

$$\begin{aligned} x + y + z &= 26 \\ 5x + 8y + 20z &= 226 \\ x + y + 3z &= 34 \end{aligned} \quad \rightarrow \quad \begin{array}{ccc|c} x & y & z & \text{total} \\ \hline 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 1 & 1 & 3 & 34 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 1 & 1 & 3 & 34 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 26 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 26 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 26 \\ 0 & 3 & 15 & 96 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 26 \\ 0 & 1 & 5 & 32 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

subtract row 1 from row 3
divide row 3 by 2
subtract 5 times row 1 from row 2
divide row 2 by 3

$$226 - 5 \times 26 = 226 - 130 = 96$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 26 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 14 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad \text{i.e.} \quad \begin{aligned} x &= 10 \\ y &= 12 \\ z &= 4 \end{aligned}$$

subtract 5 times row 3 from row 2
subtract row 2 from row 1
subtract row 3 from row 1

Example: Find all (x, y) such that $5x + 3y = 25$ and $2x - 7y = -31$.

$$\begin{bmatrix} 5 & 3 & 25 \\ 2 & -7 & -31 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{5} & 5 \\ 2 & -7 & -31 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{5} & 5 \\ 0 & -\frac{41}{5} & -41 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{5} & 5 \\ 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

divide row 1 by 5
subtract 2 times row 1 from row 2
multiply row 2 by $-\frac{5}{41}$
subtract $\frac{3}{5}$ times row 2 from row 1

$$-7 - \frac{6}{5} = -\frac{35}{5} - \frac{6}{5} = -\frac{41}{5}$$

$$-31 - 10 = -41$$

Solution: $(x, y) = (2, 5)$.

Alternatively:

$$\begin{bmatrix} 5 & 3 & 25 \\ 2 & -7 & -31 \end{bmatrix} \sim \begin{bmatrix} 2 & -7 & -31 \\ 5 & 3 & 25 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{7}{2} & -\frac{31}{2} \\ 5 & 3 & 25 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

interchange rows 1 and 2

Even better: $\begin{bmatrix} 5 & 3 & 25 \\ 2 & -7 & -31 \end{bmatrix} \sim \begin{bmatrix} 1 & 17 & 87 \\ 2 & -7 & -31 \end{bmatrix} \sim \begin{bmatrix} 1 & 17 & 87 \\ 0 & -41 & -205 \end{bmatrix} \sim \begin{bmatrix} 1 & 17 & 87 \\ 0 & 1 & 5 \end{bmatrix}$

subtract ^{2 times} row 2 from row 1 subtract 2 times row 1 from row 2 divide row 2 by -41

$$\begin{aligned} & -31 - 2 \times 87 \\ & = -31 - 174 \\ & = -205 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

Solution: $(x, y) = (2, 5)$.

subtract 17 times row 2 from row 1

check! $5 \times 2 + 3 \times 5 = 25$
 $2 \times 2 - 7 \times 5 = -31$

Elementary row operations:

- (i) add a multiple of one row to another
- (ii) multiply a row by a nonzero constant
- (iii) interchange two rows

$A \sim B$ means that A, B are linear systems having the same solutions.

We use Gaussian elimination to reduce $A_1 \sim A_2 \sim \dots \sim A_n$ where A_i represents the linear system and A_n represents an equivalent linear system (i.e. having the same solutions) but A_n is simpler than A_1 . Each step $A_i \sim A_{i+1}$ is obtained by one elementary row operation.

Why just one operation at a time?

$$\begin{cases} 5x + 3y = 25 \\ 2x - 7y = -31 \end{cases}$$

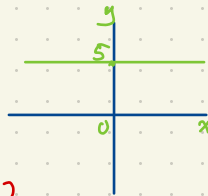
$$\begin{bmatrix} 5 & 3 & 25 \\ 2 & -7 & -31 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{5} & 5 \\ 2 & -7 & -31 \end{bmatrix}$$

divide row 1 by 5
divide row 2 by 2

$$\begin{bmatrix} 1 & \frac{3}{5} & 5 \\ 0 & -\frac{21}{5} & -\frac{41}{5} \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{5} & 5 \\ 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 5 \end{bmatrix}$$

subtract row 2 from row 1
subtract row 1 from row 2

i.e. $y = 5$
 $0 = 0$



unique solution $(2, 5)$

infinitely many solutions

Gauss



Gaussian distribution

$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 7 \\ 0 & 0 & 9 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$ are examples of matrices in reduced row echelon form:
 they cannot be simplified any further by elementary row operations.

$\begin{bmatrix} 1 & 17 & 87 \\ 0 & 1 & 5 \end{bmatrix}$ is almost reduced; it is in row echelon form.

For a linear system whose matrix is in row echelon form, we can solve for the unknowns x_1, x_2, \dots, x_n , we solve for x_n , then x_{n-1} , then x_{n-2}, \dots, x_1 by back-substitution.

eg. $\begin{bmatrix} 5 & 3 & 7 & 3 \\ 0 & 2 & 11 & 4 \\ 0 & 0 & 6 & 8 \end{bmatrix}$ is in row echelon form.

Every linear system has a unique reduced row echelon form.

In any $m \times n$ matrix, a pivot is the first nonzero entry in its row.
 (Pivots are highlighted above.)

In order for a matrix to be in row echelon form, we must have

- pivots in any row must occur to the right of pivots in any previous rows;
- any zero rows occur at the bottom.

Assuming a matrix is already in row echelon form, then to be in reduced row echelon form, we must have

- every pivot entry must be a 1
- every column having a pivot has only one nonzero entry.

Example: Solve the following linear system of 3 equations in 5 unknowns:

$$\begin{cases} x_1 + 4x_2 - x_3 + 2x_4 + 3x_5 = 6 \\ 2x_1 + 8x_2 - x_3 + 7x_4 + 4x_5 = 19 \\ -x_1 - 4x_4 + 4x_3 + 8x_4 - 4x_5 = 26 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 2 & 8 & -1 & 7 & 4 & 19 \\ -1 & -4 & 4 & 8 & -4 & 26 \end{array} \right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ -1 & -4 & 4 & 8 & -4 & 26 \end{array} \right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 3 & 10 & -1 & 32 \end{array} \right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 5 & 5 & 11 \end{array} \right]$$

This matrix is in row echelon form. This can be used to solve the linear system by back-substitution.

$$x_4 + 5x_5 = 11 \quad x_5 = t \text{ is a free parameter.}$$

$$x_4 = 11 - 5t$$

$$x_3 + 3x_4 - 2x_5 = 7$$

$$x_3 = 7 - 3x_4 + 2x_5 = 7 - 3(11 - 5t) + 2t = -26 + 17t$$

$$x_1 + 4x_2 - x_3 + 2x_4 + 3x_5 = 6$$

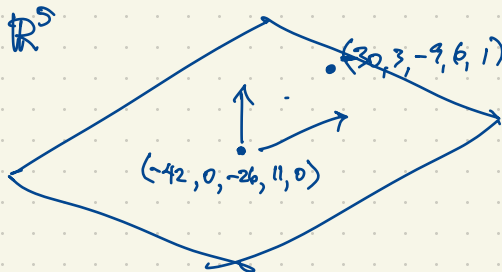
$$x_2 = s \text{ is another free parameter}$$

$$x_1 = 6 - 4x_2 + x_3 - 2x_4 - 3x_5 = 6 - 4s + (-26 + 17t) - 2(11 - 5t) - 3t = -42 - 4s + 24t$$

Solutions: $(x_1, x_2, x_3, x_4, x_5) = (-42 - 4s + 24t, s, -26 + 17t, 11 - 5t, t)$ where s, t are arbitrary.

Geometrically, the set of solutions forms a plane (2-dimensional surface) in \mathbb{R}^5 .

two parameters s, t are coordinates for the plane



Solution set inside \mathbb{R}^5 .

The point corresponding to $(s, t) = (3, 1)$ is $(30, 3, -9, 6, 1)$ is another solution.

Our system is consistent but the solution is not unique.

$$\left[\begin{array}{cccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 4 & 0 & 5 & 1 & 13 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 4 & 0 & 0 & -24 & -42 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 4 & 0 & 0 & -24 & -42 \\ 0 & 0 & 1 & 0 & -17 & -26 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array} \right]$$

(row echelon form) (reduced row echelon form)

To solve a linear system in reduced row echelon form, introduce parameters for the free variables (the variables whose columns do not contain a pivot).

In the example above, x_2 and x_5 are the free variables. Introduce s, t . $x_2 = s, x_5 = t$ can be chosen freely. Solve for the variables x_1, x_3, x_4 using the equations appearing in the reduced row echelon form:

$$\left. \begin{array}{l} x_1 + 4s - 2t = -42 \\ x_3 - 17t = -26 \\ x_4 + 5t = 11 \end{array} \right\} \Rightarrow (x_1, x_2, x_3, x_4, x_5) = (-42 - 4s + 2t, s, -26 + 17t, 11 - 5t, t)$$

where s, t are arbitrary parameters. (This is the parametric solution in terms of the parameters s, t . The system is consistent, having infinitely many solutions.)

As long as the rightmost column has no pivot, the system is consistent.

The general solution can be written as

$$\begin{aligned} (x_1, x_2, x_3, x_4, x_5) &= (-42 - 4s + 2t, s, -26 + 17t, 11 - 5t, t) \\ &= (-42, 0, -26, 11, 0) + s(-4, 1, 0, 0, 0) + t(2, 0, 17, -5, 1) \end{aligned}$$

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \quad (\text{vector addition})$$

$$c(a_1, a_2, \dots, a_n) = (ca_1, ca_2, \dots, ca_n) \quad (\text{scalar multiplication})$$

↑ scalar ↑ vector

Algebraic operations for matrices

If $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix}$

then $BA = \begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \end{bmatrix} = \begin{bmatrix} 13 & 11 & 41 \\ -1 & -27 & 23 \end{bmatrix}$

$\underbrace{\hspace{2cm}}_{2 \times 2} \underbrace{\hspace{2cm}}_{2 \times 3} \underbrace{\hspace{2cm}}_{2 \times 3}$

Here AB is undefined.

An $m \times n$ matrix has the form

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}$$

$a_{i,j}$ is the (i,j) -entry of the matrix A .

$i \in \{1, 2, \dots, m\}$

$j \in \{1, 2, \dots, n\}$.

Often $a_{i,j}$ is written a_{ij} .
(unless this results in confusion)

If A is $m \times n$ and B is $n \times r$ then AB is $m \times r$.

We can't multiply two matrices unless the number of columns in the first matrix equals the number of rows in the second matrix.

eg. $\underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \end{bmatrix}}_{2 \times 3} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}}_{3 \times 2} = \begin{bmatrix} 12 & 8 \\ 23 & 4 \end{bmatrix}$ whereas $\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}}_{3 \times 2} \underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \end{bmatrix}}_{2 \times 3} = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \\ 5 & -1 & 21 \end{bmatrix}$

If $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$ then $A^2 = \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}}_A = \begin{bmatrix} 7 & 3 \\ 1 & 4 \end{bmatrix}$, $A^3 = A^2 A = \begin{bmatrix} 7 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 17 & 18 \\ 6 & -1 \end{bmatrix}$

Recall: the linear system $\begin{cases} 5x + 3y = 25 \\ 2x - 7y = -31 \end{cases}$ has a unique solution $(x, y) = (2, 5)$.

One way to solve this: Write the linear system as $AV = b$ where A is a 2×2 matrix, $V = \begin{bmatrix} x \\ y \end{bmatrix}$ is a 2×1 matrix (i.e. column vector of length 2) and $b = \begin{bmatrix} 25 \\ -31 \end{bmatrix}$ is a 2×1 matrix of constants.

Here $A = \begin{bmatrix} 5 & 3 \\ 2 & -7 \end{bmatrix}$.

$$AV = b \text{ says } \underbrace{\begin{bmatrix} 5 & 3 \\ 2 & -7 \end{bmatrix}}_{2 \times 2} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{2 \times 1} = \underbrace{\begin{bmatrix} 25 \\ -31 \end{bmatrix}}_{2 \times 1} \text{ i.e. } \begin{cases} 5x + 3y = 25 \\ 2x - 7y = -31 \end{cases}$$

Compare: To solve $3x = 5$, multiply both sides by $3^{-1} = \frac{1}{3}$ on the left: $3^{-1}3x = 3^{-1}5$ i.e. $x = \frac{5}{3}$.

To solve $AV = b$, multiply both sides on the left by $A^{-1} = \frac{1}{41} \begin{bmatrix} 7 & 3 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} \frac{7}{41} & \frac{3}{41} \\ \frac{2}{41} & -\frac{5}{41} \end{bmatrix}$

$$AV = b \\ A^{-1}AV = A^{-1}b$$

$$\frac{1}{41} \begin{bmatrix} 7 & 3 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 2 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{41} \begin{bmatrix} 7 & 3 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} 25 \\ -31 \end{bmatrix}$$

$$\frac{1}{41} \begin{bmatrix} 41 & 0 \\ 0 & 41 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{41} \begin{bmatrix} 82 \\ 205 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_2} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

I_2 is the 2×2 identity matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \\ (n \times n \text{ identity matrix})$$

If $\underbrace{\underbrace{(ABC)}_{2 \times 7 \times 3 \times 3 \times 5}}_{2 \times 3} = \underbrace{\underbrace{A(BC)}_{2 \times 7 \times 7 \times 5}}_{2 \times 5}$ by associativity, you can do the first way since that is faster.

We say A and B commute if $AB = BA$.
 Which 2×2 matrices commute with $A = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$? Answer by solving the appropriate linear system of 4 equations in 4 unknowns. Let $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. In order for $AB = BA$ we require

$$\begin{bmatrix} 3x+z & 3y+w \\ 4z & 4w \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3x & x+y \\ 3z & z+4w \end{bmatrix}$$

$$\text{i.e. } \begin{cases} 3x+z = 3x \\ 3y+w = x+4y \\ 4z = 3z \\ 4w = z+4w \end{cases}$$

$$\begin{bmatrix} x & y & z & w & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x, z are basic variables (i.e. the pivot columns)
 y, w are free variables (non-pivot columns)

Introduce s, t as parameters. $y = s, w = t$
 and solve for x, z : $x = -s + t, z = 0$ so $B = \begin{bmatrix} -s+t & s \\ 0 & t \end{bmatrix} = s \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Check: If $s=1, t=0$ then $B = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$ and $BA = AB$.
 (a linear combination of $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}$$

$$IA = A = AI$$

$$\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}$$

For Friday's Quiz: Representation of linear systems in matrix form. $Ax = b$
 Particular and general solutions
 Null vectors
 Homogeneous linear systems $Ax = 0$ ← homogenize

$\text{Nul}(A) = \left\{ \underbrace{x \in \mathbb{R}^n}_{n \times 1} : \underbrace{Ax}_{n \times n \cdot n \times 1} = \underbrace{0}_{n \times 1} \in \mathbb{R}^n \right\}$ is the null space of A. Its vectors are called null vectors.

$$\begin{cases} x_1 + 4x_2 - x_3 + 2x_4 + 3x_5 = 6 \\ 2x_1 + 8x_2 - x_3 + 7x_4 + 4x_5 = 19 \\ -x_1 - 4x_4 + 4x_3 + 8x_4 - 4x_5 = 26 \end{cases}$$

$$\begin{bmatrix} 1 & 4 & -1 & 2 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 11 \end{bmatrix}$$

3x5

5x1

Every linear system has the form $Ax=b$ (for m linear equations in n unknowns).

$$A = \begin{bmatrix} 1 & 4 & -1 & 2 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 7 \\ 11 \end{bmatrix}$$

Some prefer to write \vec{x} or \vec{x} instead of x .

or even \mathbf{x} (bold face) or \underline{x} . For us, context is used to determine whether we are talking about a matrix, a vector, a scalar, a set, a linear transformation, a vector space, etc.

Some linear systems are inconsistent (meaning that they have no solutions). If a linear system $Ax=b$ is consistent then its solutions have the form $x = a + c_1v_1 + c_2v_2 + \dots + c_kv_k$ for some particular solution $x=a$; c_1, \dots, c_k scalars (constants)

and v_1, \dots, v_k are independent solutions of $Ax=0$.

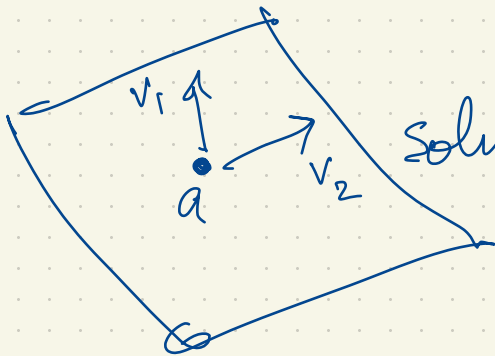
The solutions of the example $Ax=b$ above have the form $x =$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -12 \\ 0 \\ -26 \\ 11 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 24 \\ 0 \\ 17 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 - 1s + 24t \\ 0 + s \\ -26 + 17t \\ 11 - 5t \\ 0 + t \end{bmatrix} = \begin{bmatrix} -12 - 1s + 24t \\ s \\ -26 + 17t \\ 11 - 5t \\ t \end{bmatrix}$$

\uparrow a $\underbrace{\quad}_{v_1}$ $\underbrace{\quad}_{v_2}$ $\underbrace{\quad}_{5 \times 1}$

a particular solution

The general solution

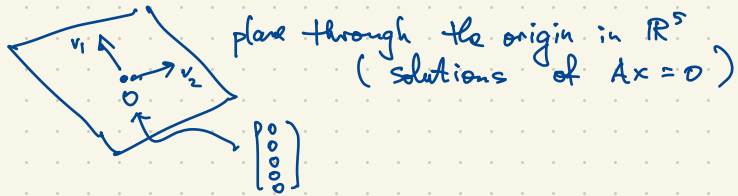


Solution set (i.e. the set of all solutions)

$A = \begin{bmatrix} 1 & 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$ gives rise to a homogeneous linear system $Ax=0$

3×1 i.e. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ " $\begin{bmatrix} -t_2 \\ \vdots \end{bmatrix}$
 (no 9 here)

i.e. $\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \end{array} \right]$ has solutions $x = sv_1 + tv_2$



Systems of the form $Ax=0$ are called homogeneous meaning if u and v are solutions
 (i.e. $Au=0$ and $Av=0$ then $A(su+tv)=0$ then so is $su+tv$.)
 $sAu + tAv = 0$

A homogeneous system $Ax=0$ is always consistent, since the zero vector $x=0$ is a solution.

$\text{Nul}(A) = \text{null space of } A = \left\{ \text{all solutions of the homogeneous system } Ax=0 \right\} \subseteq \mathbb{R}^n$

$\text{Nul}(A)$ might be: the origin in \mathbb{R}^n ,
 or maybe a line through the origin in \mathbb{R}^n ,
 plane

$$\begin{array}{c}
 \begin{matrix} \xrightarrow{m \times n} \\ \xrightarrow{n \times 1} \\ \xrightarrow{m \times 1} \end{matrix} \\
 x \in \mathbb{R}^n \quad \uparrow \quad \uparrow \\
 0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^m
 \end{array}$$

Checking answers: If we reduce $A \sim A'$, how can we check our work? (A' could be a row echelon form for A , or maybe a reduced row echelon form for A). A and A' have the same null vectors.

eg. $A = \begin{bmatrix} 1 & 4 & -1 & 2 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \sim A' = \begin{bmatrix} 1 & 4 & 0 & 0 & -29 \\ 0 & 0 & 1 & 0 & -17 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$

A' has null vectors $\begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 29 \\ 0 \\ 17 \\ -5 \\ 1 \end{bmatrix}$. Check that they are also null vectors for A .

$$\begin{bmatrix} 1 & 4 & 0 & 0 & -29 \\ 0 & 0 & 1 & 0 & -17 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 29 \\ 0 \\ 17 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Every null vector for B is also a null vector for AB .

If $Bx = 0$ then $ABx = A0 = 0$

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1/3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Last week's quiz:

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}}_A \underbrace{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix}}_B = \begin{bmatrix} 3 & 2 & 5 \\ 0 & 1 & 2 \\ 9 & 8 & 19 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix}}_B \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}}_A = \begin{bmatrix} 5 & 6 \\ 12 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 5 \\ 0 & 1 & 2 \\ 9 & 8 & 19 \end{bmatrix} \begin{bmatrix} -1/3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If C is an $n \times n$ matrix, the trace of C is the sum of the n entries on the main diagonal of C . (denoted $\text{tr} C$)

If A is $m \times n$ and B is $n \times m$, then $\text{tr}(AB) = \text{tr}(BA)$. = trace

Ex. Solve $y'' + y = x^2$.

A particular solution is $y = x^2 - 2$. Check: $y'' + y = 2 + (x^2 - 2) = x^2$.

The general solution is $y = x^2 - 2 + a \sin x + b \cos x$ where $a, b \in \mathbb{R}$ are arbitrary.

The homogenized equation $y'' + y = 0$ is homogeneous (so if y_1 and y_2 are two solutions, then a linear combination $a y_1 + b y_2$ is also a solution). The general solution of $y'' + y = 0$ is $y = a \sin x + b \cos x$.
What about $\sin(10^\circ + x) = (\cos 10^\circ) \sin x + (\sin 10^\circ) \cos x$. (Every solution of $y'' + y = 0$ is a linear combination of $\sin x$ and $\cos x$.)

A linear combination of vectors v_1, \dots, v_k is a vector of the form $c_1 v_1 + c_2 v_2 + \dots + c_k v_k$ where c_1, \dots, c_k are scalars. Moreover, $\sin x$ and $\cos x$ are linearly independent. We cannot express the general solution of $y'' + y = 0$ as a linear combination of fewer than two basic solutions.

It is correct to say that the general solution of $y'' + y = 0$ is a linear combination of $\sin x$, $\cos x$ and $\sin(10^\circ + x)$, i.e. every solution has the form $y = c_1 \sin x + c_2 \cos x + c_3 \sin(10^\circ + x)$. However this is the same as $y = (c_1 + c_3 \cos 10^\circ) \sin x + (c_2 + c_3 \sin 10^\circ) \cos x$.

It's also correct to say: every solution is a linear combination of $\sin x$ and $\sin(10^\circ + x)$.

e.g. $\cos x = \underbrace{\left(-\frac{\cos 10^\circ}{\sin 10^\circ}\right)}_{(-\cot 10^\circ)} \sin x + \underbrace{\left(-\frac{1}{\sin 10^\circ}\right)}_{(-\csc 10^\circ)} \sin(10^\circ + x)$

In fact any two solutions of $y'' + y = 0$ can be used to generate all the other solutions by taking linear combinations, as long as either of your two particular solutions is a scalar multiple of the other.

A list of vectors v_1, \dots, v_k is linearly independent if none of them is a linear combination of the others.

Alternatively, the only way to have $c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0$ is if $c_1 = \dots = c_k = 0$.

Eg. $\sin x, \cos x, \sin(10^\circ + x)$ are linearly dependent (not linearly independent) since

$\sin(10^\circ + x)$ is a linear combination of $\sin x$ and $\cos x$.

What does it mean for a list of vectors v_1, \dots, v_k to be linearly independent?

For $k=1$: $\{v_1\}$ is linearly independent iff $v_1 \neq 0$. Any list of vectors containing a zero vector is linearly dependent.

For $k=2$: $\{v_1, v_2\}$ is linearly independent iff $v_1 \neq 0$ and v_2 is not a scalar multiple of v_1 .

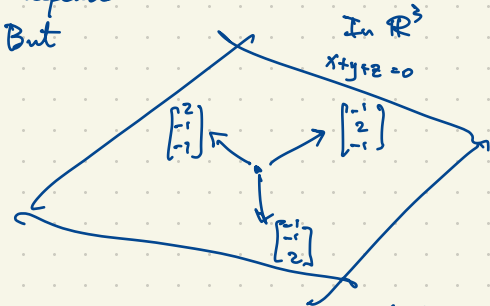
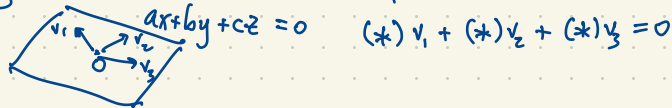
For $k=3$: $\{v_1, v_2, v_3\}$ is linearly independent iff $v_1 \neq 0$ and v_2 is not a scalar multiple of v_1 and v_3 is not a linear combination of v_1 and v_2 .

Warning: In \mathbb{R}^3 , is the set of vectors $\left\{ \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$ linearly independent?

None of these vectors is a scalar multiple of either of the others. But

$$\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ so } \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = -\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Any three vectors in the same plane are linearly dependent.



If $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ where c_1, c_2, c_3 are scalars, not all zero then v_1, v_2, v_3 are linearly dependent.

Note that if $c_3 \neq 0$ then $v_3 = -\frac{c_1}{c_3} v_1 - \frac{c_2}{c_3} v_2$ is a linear combination of v_1, v_2 .

But if $c_3 = 0$ we have $c_1 v_1 + c_2 v_2 = 0$. Here if $c_2 \neq 0$ then $v_2 = -\frac{c_1}{c_2} v_1$ is a scalar multiple of v_1 .

But if $c_2 = 0$ and $c_3 = 0$ we have $c_1 v_1 = 0$. In this case if $c_1 \neq 0$ then $v_1 = 0$ giving a linearly dependent list.

In general, when is a list of vectors v_1, \dots, v_k linearly independent? iff:

- $v_1 \neq 0$
- v_2 is not a scalar multiple of v_1
- $v_3 \dots$ linear combination of v_1, v_2

and so on. (i.e. none of the vectors in the list is a linear combination of the previous vectors in the list.)

For a matrix in row echelon form (in particular reduced row echelon form), all non-zero rows are linearly independent. Also, the pivot columns are linearly independent.

eg.
$$\begin{bmatrix} 0 & 3 & 1 & 2 & 7 & 11 & 13 & -2 \\ 0 & 0 & 0 & 5 & 6 & 4 & 7 & 6 \\ 0 & 0 & 0 & 0 & 8 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
 The first three rows are $(0, 3, 1, 2, 7, 11, 13, -2)$, $(0, 0, 0, 5, 6, 4, 7, 6)$, $(0, 0, 0, 0, 8, 4, 0)$ are linearly independent.
The four rows are linearly dependent.

The pivot columns are $\begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 5 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 11 \\ 7 \\ 8 \\ 0 \end{bmatrix}$ are linearly independent.

If A is any $m \times n$ matrix and $A \sim A'$ then every row of A' is a linear combination of rows of A (and every row of A is a linear combination of rows of A'). The same does not hold for columns

eg.
$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 2 & 0 & 6 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$