Linear Algebra

Book 2

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$f(x,y) = (3x+2y,\ x-5y)$ can be represented as a matrix transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{bmatrix} 3 & 2 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x+2y \\ x-5y \end{bmatrix}$$

Every linear operator can be expressed as matrix multiplication

eg. consider solutions of $y'' + y = 0$ i.e. $f(x) = \underbrace{a \sin x + b \cos x}_{\begin{bmatrix} a \\ b \end{bmatrix}}$

$$Df(x) = \underbrace{a \cos x - b \sin x}_{\begin{bmatrix} -b \\ a \end{bmatrix}}$$

$$D(rf + sg) = rDf + sDg$$

$$(rf+sg)' = rf' + sg'$$

$$\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{M} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$$

$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$M^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$M^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$M^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Every 2×2 real matrix A represents a linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ which is the matrix transformation $T_A \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$.

eg. $\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$ $\quad T_A$ is a counter-clockwise 90° rotation about the origin in $\mathbb{R}^2$:

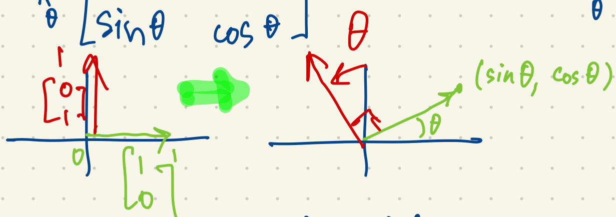
$$T_A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T_A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$T_A^4 = I \qquad I \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

A counterclockwise rotation by angle θ about the origin in $\mathbb{R}^2$ represented by the matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

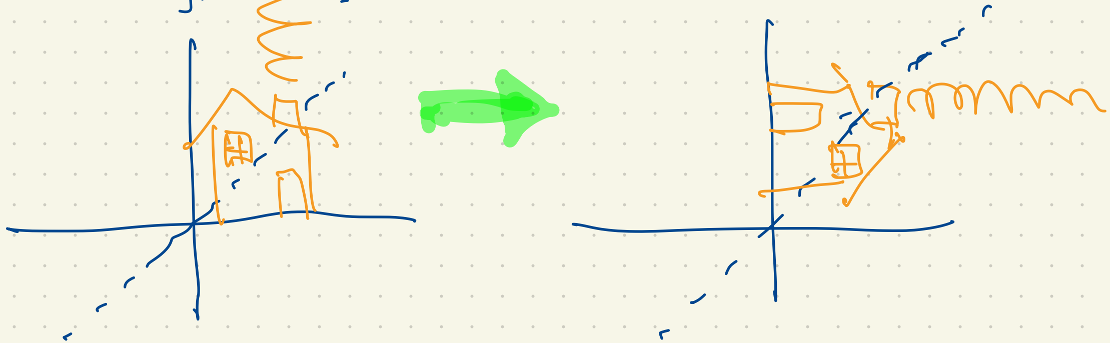
$$R_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad R_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$



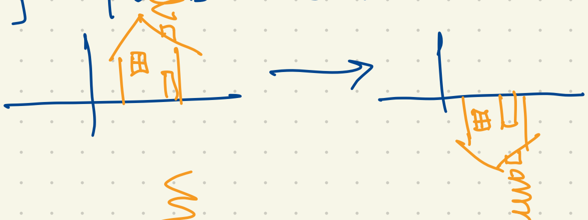
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$R_\beta R_\alpha = R_{\alpha+\beta} \qquad \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

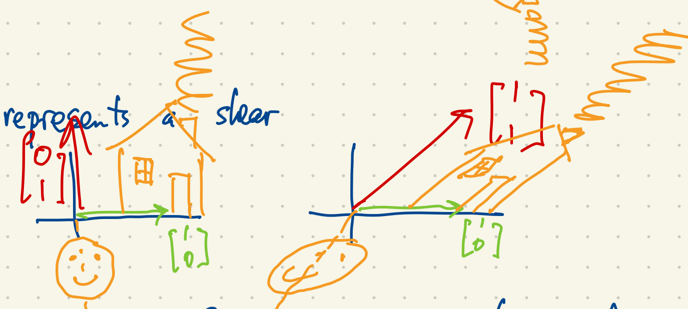
Eg. $\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}y\\x\end{bmatrix}$ is a reflection about the line $y=x$



$\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ represents a reflection in the x-axis



$\begin{bmatrix}1&1\\0&1\end{bmatrix}$ represents a shear



Every matrix transformation is a linear transformation: it takes 0 to 0 and it takes lines to lines or points. It may distort distances and angles.

Example of a "somewhat" generic linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$:

Every linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ takes 0 to 0,
takes lines to lines or points

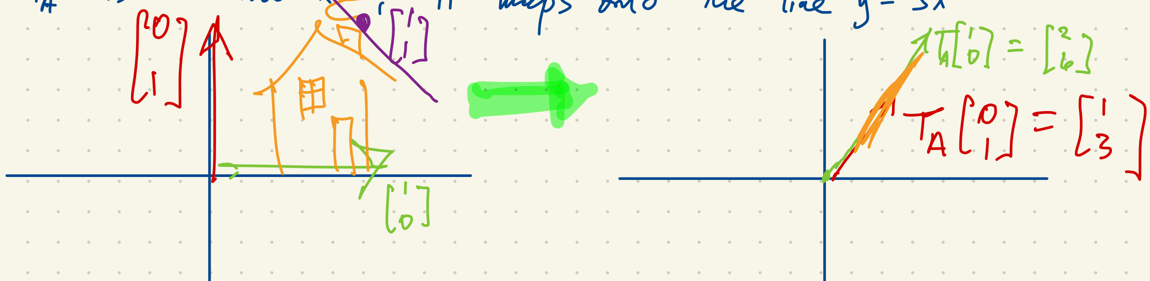
A function $f : A \to B$ is "one-to-one" if $f(x) = f(y)$ implies $x = y$. (No two inputs give the same output.)

f is "onto" if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$.

e.g. $A = \begin{bmatrix}2 & 1\\6 & 3\end{bmatrix}$ defines a linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$, $T_A\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = A\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x+y\\6x+3y\end{bmatrix}$.

This function is not one-to-one e.g. $T_A\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = T_A\left(\begin{bmatrix}-1\\5\end{bmatrix}\right) = \begin{bmatrix}3\\9\end{bmatrix}$

And T_A is not onto $\mathbb{R}^2$; it maps onto the line $y = 3x$

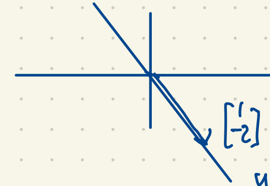


The <u>null space</u> of a linear transformation $\text{Nul}\,T = \{\underline{v} : T\underline{v} = \underline{0}\}$. (the set of null vectors of T).

Recall: $T\underline{0} = \underline{0}$

$$\text{Nul}\underbrace{\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}}_{A} = \text{Nul}\,T_A = \left\{ \begin{bmatrix} x \\ -2x \end{bmatrix} : x \in \mathbb{R} \right\}$$

$$A\begin{bmatrix} x \\ -2x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$



$\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is a null vector.

$y = -2x$

T is one-to-one iff $\text{Nul}\,T = \{\underline{0}\}$ (the only null vector is $\underline{0}$).

This statement should be clear:

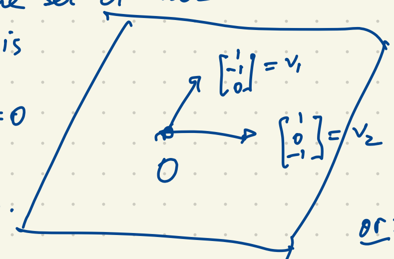
On the one hand, suppose T is one-to-one. If $\underline{v} \in \text{Nul}\,T$ then $T\underline{v} = \underline{0} = T\underline{0}$ then $\underline{v} = \underline{0}$.
This says: if T is one-to-one then $\text{Nul}\,T = \{\underline{0}\}$

Conversely, suppose $\text{Nul}\,T = \{\underline{0}\}$. If $T\underline{v} = T\underline{w}$ then $T(\underline{v}-\underline{w}) = T\underline{v} - T\underline{w} = \underline{0}$
so $\underline{v} - \underline{w} \in \text{Nul}\,T$ i.e. $\underline{v} - \underline{w} = \underline{0}$ i.e. $\underline{v} = \underline{w}$.

"<u>Span</u>" can be used as a noun or as a verb.

The <u>span</u> of a list of vectors $v_1, \dots, v_k$ is the set of all linear combinations of $v_1, \dots, v_k$.

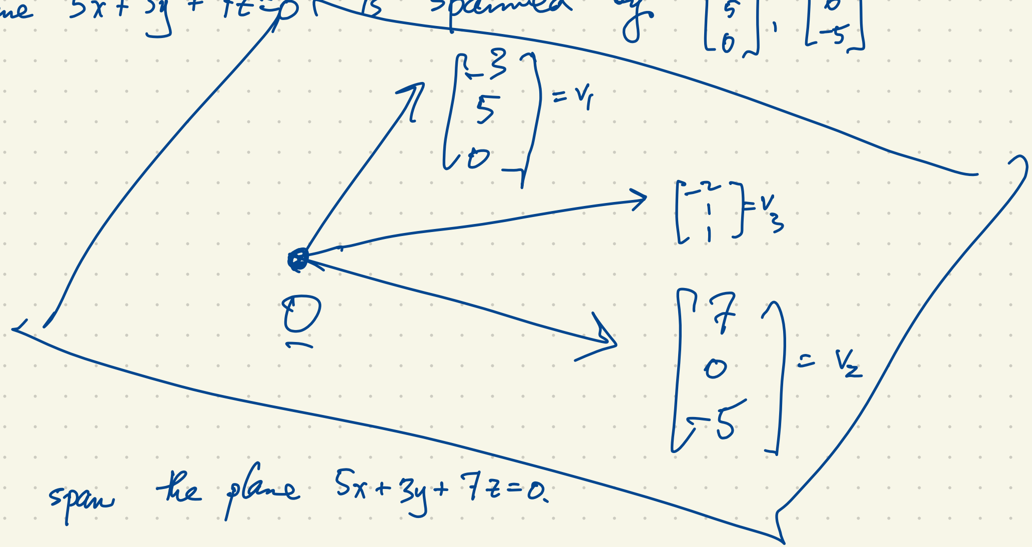
eg. the <u>span</u> of the vectors $\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ in $\mathbb{R}^3$ is
the plane $x + y + z = 0$
in $\mathbb{R}^3$
i.e. the plane $z = -x - y$.



We say that the span of v_1 and v_2 is the plane $x + y + z = 0$.

<u>or</u>: v_1 and v_2 <u>span</u> the plane $x + y + z = 0$.

eg. the plane $5x+3y+7z=0$ is spanned by $\begin{bmatrix}-3\\5\\0\end{bmatrix}, \begin{bmatrix}7\\0\\-5\end{bmatrix}$



v_1, v_2, v_3 span the plane $5x+3y+7z=0$.

Given any set of vectors $S \subset \mathbb{R}^3$, the <u>span</u> of S (denoted $\text{span}\, S = \{\text{linear combinations of vectors in } S\}$) is either $\{\underline{0}\}$ or a line through $\underline{0}$, or a plane through $\underline{0}$, or $\mathbb{R}^3$.

Friday: Quiz 5 on <u>Span</u>.

The <u>image</u> of T is $\{T_A \underline{v} : \underline{v} \in \text{domain of } T_A\}$ is the span of the columns of A.

Eg. $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ defines a linear transformation $T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

(here $\mathbb{R}^3$ consists of 3×1 column vectors)

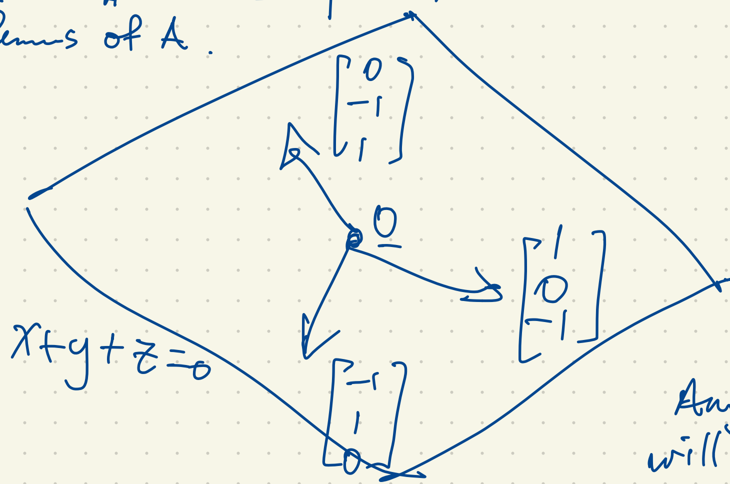
$$T_A(v) = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y-z \\ -x+z \\ x-y \end{bmatrix}$$

$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

The image of T_A is $\{T_A v : v \in \mathbb{R}^3\} = \left\{ \underbrace{\begin{bmatrix} y-z \\ -x+z \\ x-y \end{bmatrix}}_{x\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + z\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} : x, y, z \in \mathbb{R} \right\}$

(a linear combination of the columns of A)

The image of T_A is the span of the columns of A.

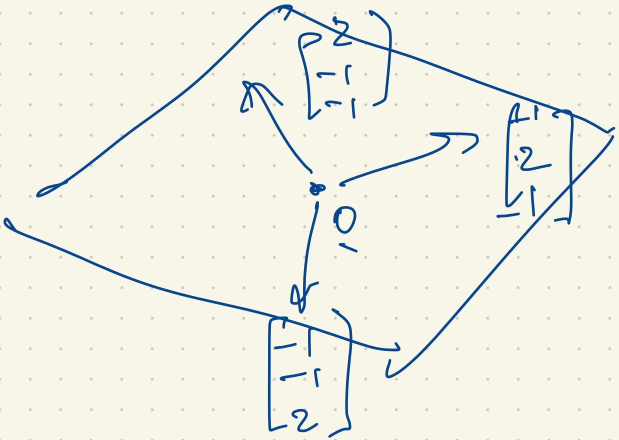


T_A is <u>not</u> onto $\mathbb{R}^3$. This happens because the columns of A fail to span $\mathbb{R}^3$.

Any 3 <u>linearly</u> independent vectors in $\mathbb{R}^3$ will span all of $\mathbb{R}^3$ (their span is $\mathbb{R}^3$).

Another example: $B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ defines a linear transformation $T_B : \mathbb{R}^3 \to \mathbb{R}^3$.

Once again T_B is not onto $\mathbb{R}^3$; its image is the span of the columns of B i.e. the plane $x+y+z=0$ through the origin in $\mathbb{R}^3$.

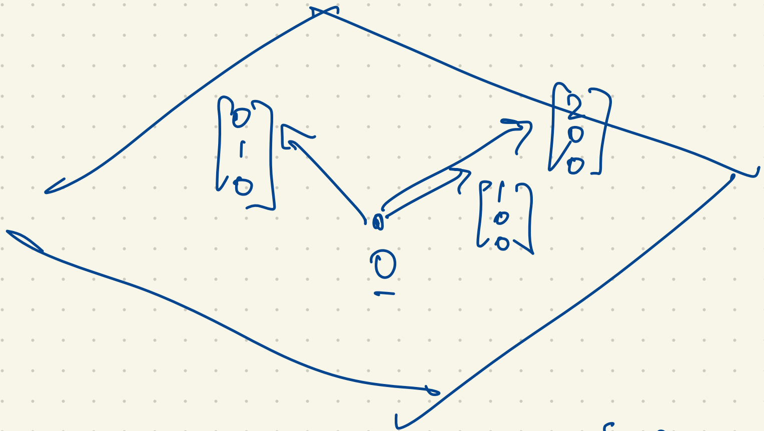


$C = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ has three linearly independent columns spanning $\mathbb{R}^3$ i.e. the image of T_C is $\mathbb{R}^3$ i.e. T_C is onto $\mathbb{R}^3$.

Check: If $a\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + c\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3a-b-c \\ -a+2b-c \\ -a-b+2c \end{bmatrix}$

$A = \begin{bmatrix}1&2&0\\0&0&1\\0&0&0\end{bmatrix}$ has $\left\{\begin{bmatrix}x\\y\\0\end{bmatrix} : x,y\in\mathbb{R}\right\}$ as the span of its columns.

T_A is not onto.



The span of the <u>rows</u> of A is $\{ [a, 2a, b] : a,b\in\mathbb{R}\}$

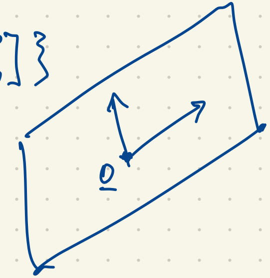
A <u>subspace</u> of $\mathbb{R}^n$ generalizes the notion of $\{\underline{0}\}$, line through the origin, plane through the origin, etc. up to and including $\mathbb{R}^n$ itself. The dimension of such a subspace is $0,1,2,3,\dots,n$.

Given any set $S\subset\mathbb{R}^n$ (any set of vectors) then span S = {linear combinations of vectors in S} is a subspace of $\mathbb{R}^n$. Another way is to solve any homogeneous linear system in n variables. The latter case is the same thing as finding the null space of a linear transformation.

In particular if A is an $m\times n$ matrix then $\mathrm{Nul}\,A = \{\underline{v}\in\mathbb{R}^n : A\underline{v} = \underset{\text{in }\mathbb{R}^m}{\underline{0}}\}$ is a subspace of $\mathbb{R}^n$.

Eg. a 2-dimensional subspace of $\mathbb{R}^3$ (i.e. a plane through the origin) can be described in either of two ways.

$U = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \right\}$



$x + 3y - z = 0$

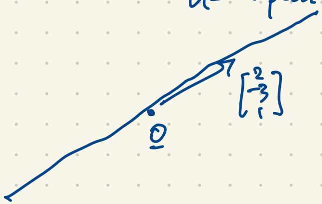
Alternatively, $U = \operatorname{Nul}\begin{bmatrix} 1 & 3 & -1 \end{bmatrix}$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \right\}$$

$$= \left\{ s\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

Eg. a 1-dimensional subspace of $\mathbb{R}^3$ (i.e. a line through the origin).

$U = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right\}$



$U = \operatorname{Nul}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

i.e. $\begin{cases} x + y + z = 0 \\ x + 2y + 4z = 0 \end{cases}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

$U = \operatorname{Nul}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ x, y are basic variables; z is a free variable.

$z = t$ where t is arbitrary; solve for y, x

$y = -3t$

$x = 2t$

$$U = \left\{ \begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\{ t\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

The solutions of $y''+y=0$ form a vector space $\{y : y''+y=0\} = \text{span}\{\sin x, \cos x\}$

$= \{a\sin x + b\cos x : a,b\in\mathbb{R}\}$

$= \text{Nul}\, T.$

Here $Ty = y''+y$ is a function mapping one function to another.

$$T: \{\text{functions}\} \longrightarrow \{\text{functions}\}$$

T is a linear transformation since $T(ay_1+by_2) = aTy_1 + bTy_2$.

Let $T: V \to W$ be a linear transformation.

T is one-to-one iff $\text{Nul}\, T = 0$.

T is onto iff every $w\in W$ has the form $w = Tv$ for some $v\in V$.

T is bijective iff it is both one-to-one and onto. Such functions T have an inverse T^{-1}.

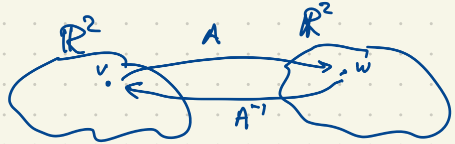
T^{-1} must also be linear.

Eg. consider the 2×2 matrix $A = \begin{bmatrix}3 & 2\\ 8 & 5\end{bmatrix}$ which represents a linear transformation $T_A: \mathbb{R}^2 \to \mathbb{R}^2$.

Find the inverse matrix A^{-1}.

$A^{-1}(Av) = v$ $\qquad$ $A(A^{-1}w) = w$

$A^{-1}A = I$ $\qquad$ $AA^{-1} = I$



$I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ identity

Fri. Oct 13 Quiz: Inverses of Matrices

A 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $ad-bc \neq 0$, in which case $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Eg. for $A = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$ we have $3\cdot 5 - 2\cdot 8 = -1$, $A^{-1} = \frac{1}{-1}\begin{bmatrix} 5 & -2 \\ -8 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 8 & -3 \end{bmatrix}$.

Check: $AA^{-1} = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^{-1}A = I$.

Eg. $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$. Compute B^{-1}.

General method: To compute A^{-1}, if it exists, write down $\overbrace{[\underbrace{A}_{n\times n} \mid I_n]}^{n\times 2n}$, $I_n = \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix}$, and row reduce leading to $\underbrace{[I \mid A^{-1}]}_{n\times 2n}$

In our case

$$[B \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 2 & 8 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 2 & 8 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -\frac{5}{2} & 4 & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right]$$

$$B^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

Check: $B^{-1}B = \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

If the pivots are not all in the leftmost n columns, we don't get I_n on the left. In this case A is not invertible.

Eg. $A = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$.

$$[A \mid I] = \left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 8 & 5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ -1 & -1 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 1 & 1 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 0 & -1 & -8 & 3 \\ 1 & 1 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 3 & -1 \\ 0 & -1 & -8 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 1 & 3 & -1 \\ 0 & 1 & 8 & -3 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 8 & -3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -5 & 2 \\ 8 & -3 \end{bmatrix}$$

Eg. $A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$ has $3 \cdot 2 - 1 \cdot 6 = 0$ so A is not invertible. What goes wrong in our algorithm?

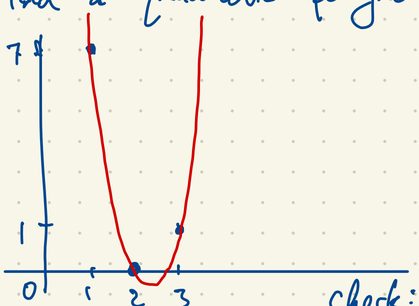
$$[A \mid I] = \left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 6 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right]$$

The pivots do not appear in the leftmost two columns so we conclude that A is not invertible. The image of T_A is the span of the columns of A, namely $\operatorname{span}\left\{\begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$, not $\mathbb{R}^2$. So T_A is not invertible i.e. A is not invertible.

Eg. Find a quadratic polynomial $f(t) = at^2 + bt + c$ having table of values

t	$f(t)$
1	7
2	0
3	1

$$= c + bt + at^2$$



$$f(1) = c + b + a = 7$$
$$f(2) = c + 2b + 4a = 0$$
$$f(3) = c + 3b + 9a = 1$$

Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ -19 \\ 4 \end{bmatrix}$$

So $f(t) = 22 - 19t + 4t^2$

Check: $f(1) = 7$, $f(2) = 0$, $f(3) = 1$ ✓

The solution of a linear system $A\underline{x} = \underline{b}$ is $\underline{x} = A^{-1}\underline{b}$ assuming A is an invertible $n \times n$ matrix.

$$[A \mid I] \sim \cdots \sim [I \mid A^{-1}]$$

$A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$ is not invertible since the span of its columns is $\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ ie. A has linearly dependent columns. $\begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Alternatively, A has a null vector $\begin{bmatrix} 1 \\ -3 \end{bmatrix} \in \text{Nul } A$ since $A\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = A\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\text{Nul } A = \text{span}\left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$ so A is not one-to-one.

The linear system $Ax = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has many solutions.

The linear system $Ax = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ has no solutions. since $\begin{bmatrix} 1 \\ 7 \end{bmatrix} \notin \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

In 5th edition, I'm omitting
- 2.4 Partitioned Matrices
- 2.5 Matrix Factorizations
- 2.6 Leontief Input/Output Model
- 2.7 Computer graphics

Continue with 2.8 : Subspaces of $\mathbb{R}^n$

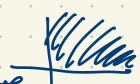
A subspace of $\mathbb{R}^n$ is a subset $U \subseteq \mathbb{R}^n$ such that

(i) $\underline{0} \in U$

(ii) For all $u, v \in U$, $u + v \in U$.

(iii) For all $u \in U$ and scalar $c \in \mathbb{R}$, $cu \in U$.

Eg. In $\mathbb{R}^2$, $\{(x,y) : x, y \geq 0\}$ is not a subspace.



Think of: $\{0\}$, line through the origin, plane through the origin, etc.

$U_1 \cap U_2 = \{u : u \in U_1 \text{ and } u \in U_2\}$

If U_1, U_2 are subspaces of $\mathbb{R}^n$, is $U_1 \cap U_2$ also a subspace of $\mathbb{R}^n$?

(i) Since $0 \in U_1$ and $0 \in U_2$, $0 \in U_1 \cap U_2$.

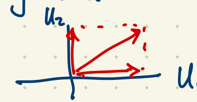
(ii) Let $u, v \in U_1 \cap U_2$. Then $u + v \in U_1$ and $u + v \in U_2$ so $u + v \in U_1 \cap U_2$.

(iii) Let c be a scalar and $u \in U_1 \cap U_2$. Then $cu \in U_1$ and $cu \in U_2$ so $cu \in U_1 \cap U_2$.

So yes, the intersection of two subspaces is a subspace.

$U_1 \cup U_2 = \{u : u \in U_1 \text{ or } u \in U_2\}$ i.e. u is in at least one of U_1 or U_2, possibly both.

If U_1 and U_2 are subspaces of $\mathbb{R}^n$, must $U_1 \cup U_2$ also be a subspace? No.



eg. $U_1 = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\} =$ the x-axis in $\mathbb{R}^2$

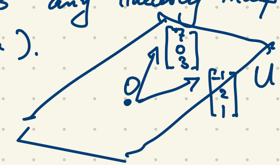
$U_2 = \text{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\} = \cdots$ y-axis in $\mathbb{R}^2$

$$\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\text{in } U_1 \cup U_2} + \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\text{in } U_1 \cup U_2} = \begin{bmatrix}1\\1\end{bmatrix} \notin U_1 \cup U_2 .$$

Alternatively, a subspace is a nonempty subset $U \subseteq \mathbb{R}^n$ such that linear combinations of vectors in U is still in U i.e. $\text{span}\, U = U$.

If U is a subspace of $\mathbb{R}^n$ $(U \leq V)$ then a <u>basis</u> for U is any linearly independent set of vectors spanning U.

eg. in $\mathbb{R}^3$, let U be the plane $3x + 5y - 7z = 0$ (through the origin).



The list of vectors $\begin{bmatrix}7\\0\\3\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}$ is a basis for U. These two vectors are linearly independent by inspection. Moreover $\text{span}\left\{\begin{bmatrix}7\\0\\3\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}\right\} = U$ (this is not quite obvious but we will soon see why it's true).

Another basis for U is $\left\{\begin{bmatrix}0\\7\\5\end{bmatrix}, \begin{bmatrix}-5\\3\\0\end{bmatrix}\right\}$

The <u>dimension</u> of U is 2 because we have a basis consisting of 2 vectors.

Syllabus

~~HW~~	~~10~~
~~Quizzes~~	~~20~~
~~Test 1~~	~~20~~
~~Test 2~~	~~20~~
~~Exam~~	~~30~~
	~~100~~

HW	30
Quizzes	20
Test	20
Exam	30
	100

I'll correct this online and email everyone with this correction.

How do we find a basis for a subspace of $\mathbb{R}^n$?

Eg. If A is an $m\times n$ matrix, $\text{Row } A = \text{span}(\text{rows of } A) \le \mathbb{R}^n$ (really $1\times n$ vectors)

$\text{Col } A = \text{Span}(\text{columns of } A) \le \mathbb{R}^m$ (really $m\times 1$ vectors)

(the <u>row space</u> and <u>column space</u> of A).

Take e.g. $A = \begin{bmatrix} 0 & 1 & -1 & 0 & 3 & 6 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ in reduced row echelon form.

Row A has basis $(0,1,-1,0,3,6)$, $(0,0,0,1,-5,2)$ so Row A is 2-dimensional : $\dim(\text{Row } A) = 2$.

The dimension of $U \le \mathbb{R}^n$ is the number of vectors in a basis for U.

Col A has basis $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$.

$$\begin{aligned}\text{Col } A &= \text{span}(\text{columns of } A) \\ &= \left\{ c_1\begin{bmatrix}0\\0\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\0\end{bmatrix} + c_3\begin{bmatrix}-1\\0\\0\end{bmatrix} + c_4\begin{bmatrix}0\\1\\0\end{bmatrix} + c_5\begin{bmatrix}3\\-5\\0\end{bmatrix} + c_6\begin{bmatrix}6\\2\\0\end{bmatrix} : c_1, c_2, \dots, c_6 \text{ any scalars} \right\} \\ &= \left\{ c_2\begin{bmatrix}1\\0\\0\end{bmatrix} + c_4\begin{bmatrix}0\\1\\0\end{bmatrix} : c_2, c_4 \text{ scalars} \right\} = \left\{ \begin{bmatrix}x\\y\\0\end{bmatrix} : x,y \in \mathbb{R} \right\} \text{ (the } xy\text{-plane)}\end{aligned}$$

$\dim \text{Col } A = 2$.

Although row vectors have length 6 and column vectors have length 3, the row space and column space have the same dimension. (equal to the number of pivots).

What if A is not in reduced row echelon form?

Eg. $B = \begin{bmatrix} 0 & 2 & -2 & 1 & 1 & 14 \\ 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 1 & -1 & -2 & 13 & 2 \end{bmatrix}$ Row $B \leq \mathbb{R}^6$, Col $B \leq \mathbb{R}^3$

$$B \sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 2 & -2 & 1 & 1 & 14 \\ 0 & 1 & -1 & -2 & 13 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 0 & 0 & -5 & 25 & -10 \\ 0 & 1 & -1 & -2 & 13 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 1 & -1 & -2 & 13 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & -5 & 25 & -10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 1 & -5 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 0 & 3 & 6 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A$$

The rank of a matrix plus the nullity of the matrix is the number of columns of the matrix.

Row B = Row A has basis $(0,1,-1,0,3,6)$, $(0,0,0,1,-5,2)$

Col $B \neq$ Col A <u>but</u> Col B has basis $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

In general, the pivot columns of A (= reduced row echelon form of B) tell us which columns of B give a basis for col B.

e.g. $\begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 1 \\ -12 \\ 13 \end{bmatrix} = (3) \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + (-5) \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

dim Row B = dim Col B = no. of pivots = rank B

dim Nul B = (no. of columns of B) − (no. of pivots)

The <u>rank</u> of a matrix is the dimension of its row and column space. The <u>nullity</u> of a matrix is the dimension of its null space.

Fact: Although Row B and Col B are very different (one is a set of 1×6 row vectors; the other is a set of 3×1 column vectors) they have the same dimension; in each case the dimension is the number of pivots of A, the reduced row-echelon form of B.

Another important subspace related to B is its null space Nul B = Nul A which has basis

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$$
$$\begin{bmatrix} 0 & 1 & -1 & 0 & 3 & 6 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

basic variables x_2, x_4
free variables x_1, x_3, x_5, x_6
Choose parameters r, s, t, u

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} r \\ s-3t-6u \\ s \\ 5t-2u \\ t \\ u \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -3 \\ 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ -6 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Another way to get a basis for the column space of B is to <u>transpose</u> the matrix B to obtain its transpose

$$B^T = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ -2 & -1 & -1 \\ 1 & 3 & -2 \\ 1 & -12 & 13 \\ 14 & 12 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\operatorname{rank} B^T = 2$

A basis for the row space of B^T is $(1,0,1)$, $(0,1,-1)$;

a basis for the column space of B^T is $\begin{bmatrix} 0 \\ 2 \\ -12 \\ 1 \\ 1 \\ 14 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 3 \\ -12 \\ 12 \end{bmatrix}$.

<u>So</u>: a basis for the column space of B is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$;

and a basis for the row space of B is $(0,2,-2,1,1,14)$, $(0,1,-1,3,-12,12)$ (the first two rows of B).

$$B = \begin{bmatrix} 0 & 2 & -2 & 1 & 1 & 14 \\ 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 1 & -1 & -2 & 13 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = (2)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -12 \\ 13 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (-12)\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

If $A = \begin{bmatrix} 5 & 3 \\ 7 & -1 \end{bmatrix}$ then $A\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 7 & -1 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 5x+3z \\ 7x-z \end{bmatrix}$ and $A\begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} 5y+3w \\ 7y-w \end{bmatrix}$ so

$$A\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 5x+3z & 5y+3w \\ 7x-z & 7y-w \end{bmatrix}$$

$$A\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 7 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x+3z & y+3w \\ z & w \end{bmatrix}$$

This matrix is an elementary matrix; it corresponds to an elementary row operation of adding 3× row 2 to row 1.

NOVEMBER 2023

SUN	MON	TUE	WED	THU	FRI	SAT
29	30 Hw2 due	31	1	2	3	4
5	6	7	8 Test	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	1	2

The three kinds of elementary row operations on an $m \times n$ matrix A correspond to left-multiplication by an $m \times m$ elementary matrix.

- Adding an entry "a" in the (i,j) position of $I_m = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ $(i \neq j)$ gives an elementary matrix $\begin{array}{c} \\ i \end{array}\begin{bmatrix} 1 & & \\ & 1 & \\ a & & \ddots \\ & & & 1 \end{bmatrix} = E$. Then EA is obtained from A by adding "a" times rows j to row i.

$$E[I_m | A] = [EI | EA] = [E | EA]$$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = E$ elementary matrix
add 2 times row 1 to row 2

eg. $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 4 & 1 & 5 \end{bmatrix}$
add 2 times row 1 to row 2

$$EA = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 4 & 1 & 5 \end{bmatrix}$$

- The row operation "multiply row 2 by 3": $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 6 & 3 & 9 \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = E$

$$EA = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 6 & 3 & 9 \end{bmatrix}$$

- The row operation "switch rows 2 and 3", $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ 5 & 1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 5 & 1 & 1 & 4 \\ 2 & 1 & 3 & 2 \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ 5 & 1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 5 & 1 & 1 & 4 \\ 2 & 1 & 3 & 2 \end{bmatrix}$$

Every invertible matrix is a product of elementary matrices. A noninvertible matrix is not a product of elementary matrices.

<u>Shoe-Sock Theorem</u> If A and B are invertible $n \times n$ matrices then AB is invertible $n \times n$. $(AB)^{-1} = B^{-1}A^{-1}$.

Check: $(A\underbrace{B)(B^{-1}}_{BB^{-1} = I_n}A^{-1}) = AI_nA^{-1} = AA^{-1} = I_n$

$(B^{-1}\underbrace{A^{-1})(A}_{A^{-1}A = I}B) = B^{-1}IB = B^{-1}B = I$

$(AB)(A^{-1}B^{-1}) = ?$ does not usually simplify.

$(AB)v = A(Bv)$

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

Every elementary row operation is invertible. In other words, elementary matrices are invertible.

If $A = E_1E_2E_3\cdots E_r$ where each E_i is an elementary $m\times m$ matrix then A is invertible and

$$A^{-1} = (E_1E_2\cdots E_r)^{-1} = E_r^{-1}E_{r-1}^{-1}\cdots E_2^{-1}E_1^{-1}$$ where $E_r^{-1},\dots,E_1^{-1}$ are again elementary matrices.

Why does our algorithm for finding A^{-1} work?

$$\underset{m\times 2m}{[A\,|\,I]} \sim E_1[A\,|\,I] = [E_1A\,|\,E_1] \sim E_2[E_1A\,|\,E_1] = [E_2E_1A\,|\,E_2E_1] \sim \cdots \sim E_r[E_{r-1}\cdots E_2E_1A\,|\,E_{r-1}E_{r-2}\cdots E_1]$$

$$[\underbrace{E_rE_{r-1}\cdots E_2E_1A}_{I}\,|\,\underbrace{E_rE_{r-1}\cdots E_2E_1}_{A^{-1}}]$$

If $\underbrace{E_rE_{r-1}\cdots E_2E_1}_{A^{-1}}A = I$ then $A^{-1}=E_rE_{r-1}\cdots E_2E_1$

$$A = E_1^{-1}E_2^{-1}\cdots E_{r-1}^{-1}E_r^{-1}$$

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$$

$$\begin{bmatrix}2&1\\3&2\end{bmatrix} = \begin{bmatrix}2&-1\\-3&2\end{bmatrix}$$

Eg. Write $A=\begin{bmatrix}2&1\\3&2\end{bmatrix}$ as a product of elementary matrices.

$$[A\,|\,I] = \left[\begin{array}{cc|cc}2&1&1&0\\3&2&0&1\end{array}\right] \underset{\begin{bmatrix}1&0\\-1&1\end{bmatrix}}{\sim} \left[\begin{array}{cc|cc}2&1&1&0\\1&1&-1&1\end{array}\right] \underset{\begin{bmatrix}0&1\\1&0\end{bmatrix}}{\sim} \left[\begin{array}{cc|cc}1&1&-1&1\\2&1&1&0\end{array}\right] \underset{\begin{bmatrix}1&0\\-2&1\end{bmatrix}}{\sim} \left[\begin{array}{cc|cc}1&1&-1&1\\0&-1&3&-2\end{array}\right] \underset{\begin{bmatrix}1&1\\0&1\end{bmatrix}}{\sim} \left[\begin{array}{cc|cc}1&0&2&-1\\0&-1&3&-2\end{array}\right] \underset{\begin{bmatrix}1&0\\0&-1\end{bmatrix}}{\sim} \left[\begin{array}{cc|cc}1&0&2&-1\\0&1&-3&2\end{array}\right]$$

$$A = \begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&0\\2&1\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}.$$