

Linear Algebra

Book 2

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$f(x, y) = (3x+2y, x-5y)$ can be represented as a matrix transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{bmatrix} 3 & 2 \\ 1 & -5 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x+2y \\ x-5y \end{pmatrix}$$

Every linear operator can be expressed as matrix multiplication

eg. consider solutions of $y''+y=0$ i.e. $f(x) = \underbrace{a \sin x + b \cos x}_{\begin{pmatrix} a \\ b \end{pmatrix}}$

$$Df(x) = \underbrace{a \cos x - b \sin x}_{\begin{pmatrix} b \\ a \end{pmatrix}}$$

$$D(rf+sg) = rDf + sDg \quad \begin{pmatrix} b \\ a \end{pmatrix}$$

$$(rf+sg)' = rf' + sg'$$

$$\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_M \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$$

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

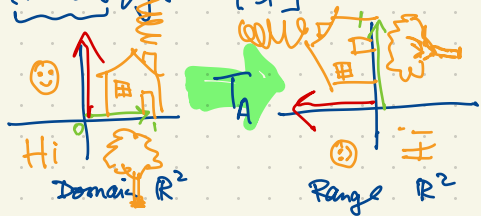
$$M^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$M^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$M^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Every 2×2 real matrix A represents a linear transformation $T_A: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which is the matrix transformation $T_A \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$.

eg. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$ T_A is a counter-clockwise 90° rotation about the origin in \mathbb{R}^2 :

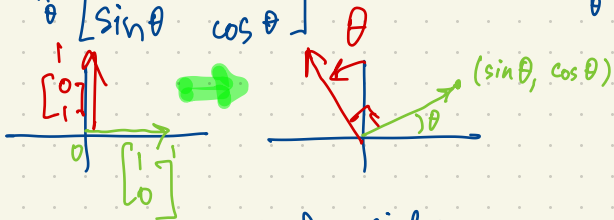


$$T_A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T_A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$T_A^{-1} = I \quad I \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

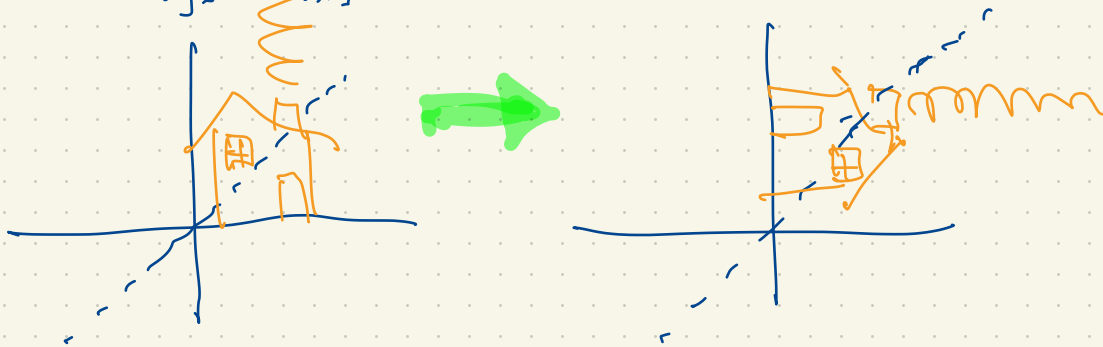
A counterclockwise rotation by angle θ about the origin in \mathbb{R}^2 represented by the matrix $R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \ \theta \end{bmatrix}$ $R_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ $R_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$



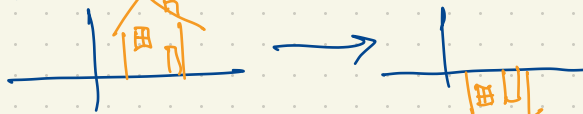
$$\begin{aligned} \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \end{aligned}$$

$$R_\beta R_\alpha = R_{\alpha + \beta} \quad \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} = \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix}$$

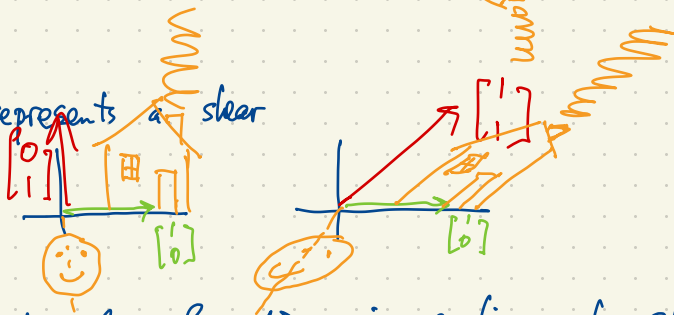
Eg. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$ is a reflection about the line $y=x$



$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ represents a reflection in the x-axis

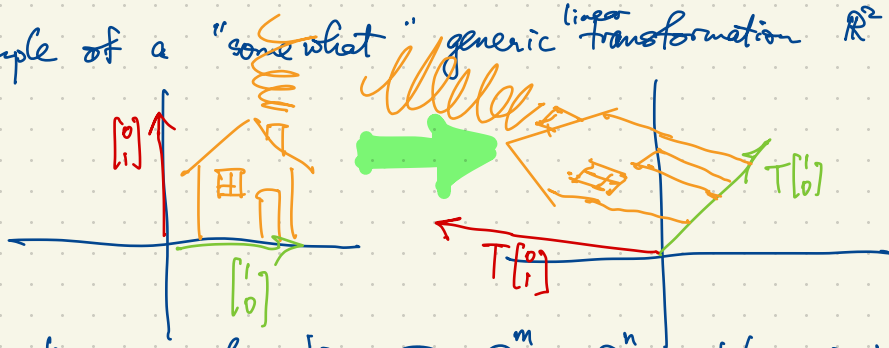


$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ represents a shear



Every matrix transformation is a linear transformation: it takes \mathbb{D} to \mathbb{D} and it takes lines to lines. It may distort distances and angles or points.

Example of a "somewhat" generic linear transformation $\mathbb{R}^2 \rightarrow \mathbb{R}^2$:



Every linear transformation $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ takes 0 to 0 ,
takes lines to lines or points

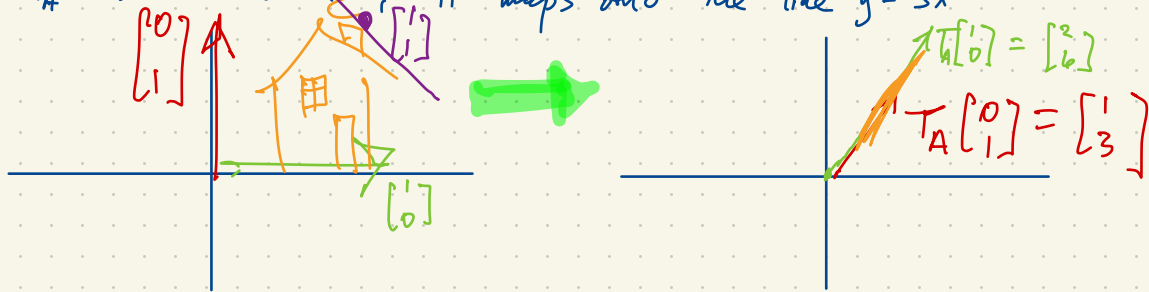
$$= \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

A function $f: A \rightarrow B$ is "one-to-one" if $f(x) = f(y)$ implies $x = y$. (No two inputs give the same output.)
 f is "onto" if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$.

eg. $A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$ defines a linear transformation $T_A: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T_A \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ 6x+3y \end{bmatrix}$.

This function is not one-to-one e.g. $T_A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = T_A \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

And T_A is not onto \mathbb{R}^2 ; it maps onto the line $y = 3x$

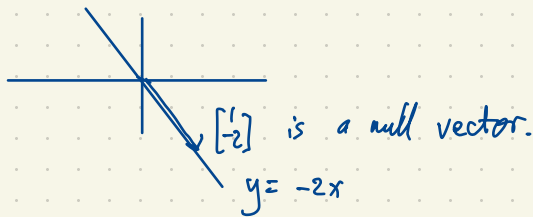


The null space of a linear transformation $\text{Nul } T = \{ \underline{v} : T\underline{v} = \underline{0} \}$. (the set of null vectors of T)

Recall: $T\underline{0} = \underline{0}$

$$\text{Nul} \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} = \text{Nul } T_A = \left\{ \begin{bmatrix} x \\ -2x \end{bmatrix} : x \in \mathbb{R} \right\}$$

$$A \begin{bmatrix} x \\ -2x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



T is one-to-one iff $\text{Nul } T = \{ \underline{0} \}$ (the only null vector is $\underline{0}$).

This statement should be clear:

On the one hand, suppose T is one-to-one.

If $\underline{v} \in \text{Nul } T$ then $T\underline{v} = \underline{0} = T\underline{0}$ then $\underline{v} = \underline{0}$.

This says: if T is one-to-one then $\text{Nul } T = \{ \underline{0} \}$

Conversely, suppose $\text{Nul } T = \{ \underline{0} \}$.

If $T\underline{v} = T\underline{w}$ then $T(\underline{v} - \underline{w}) = T\underline{v} - T\underline{w} = \underline{0}$

so $\underline{v} - \underline{w} \in \text{Nul } T$ i.e. $\underline{v} - \underline{w} = \underline{0}$ i.e. $\underline{v} = \underline{w}$.

"Span" can be used as a noun or as a verb.

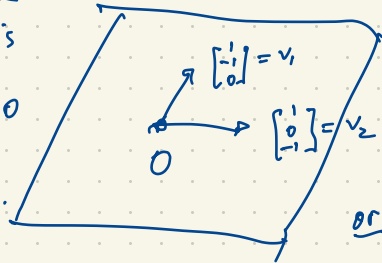
The span of a list of vectors v_1, \dots, v_k is the set of all linear combinations of v_1, \dots, v_k .

eg. the span of the vectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ in \mathbb{R}^3 is

the plane $x + y + z = 0$

in \mathbb{R}^3

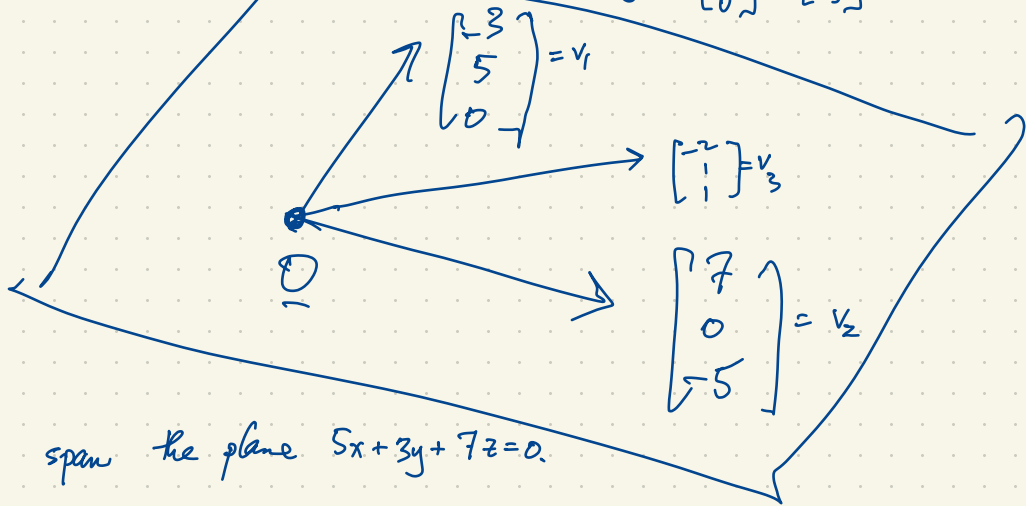
i.e. the plane $z = -x - y$.



We say that the span of v_1 and v_2 is the plane

or: v_1 and v_2 span the plane $x + y + z = 0$.

eg. the plane $5x + 3y + 7z = 0$ is spanned by $\begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$



Given any set of vectors $S \subset \mathbb{R}^3$, the span of S (denoted $\text{span } S = \{\text{linear combinations of vectors in } S\}$) is either $\{0\}$, or a line through 0 , or a plane through 0 , or \mathbb{R}^3 .

Friday: Quiz 5 on Span.

The image of T is $\{T_A \underline{v} : \underline{v} \in \text{domain of } T_A\}$ is the span of the columns of A .

Eg. $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ defines a linear transformation $T_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

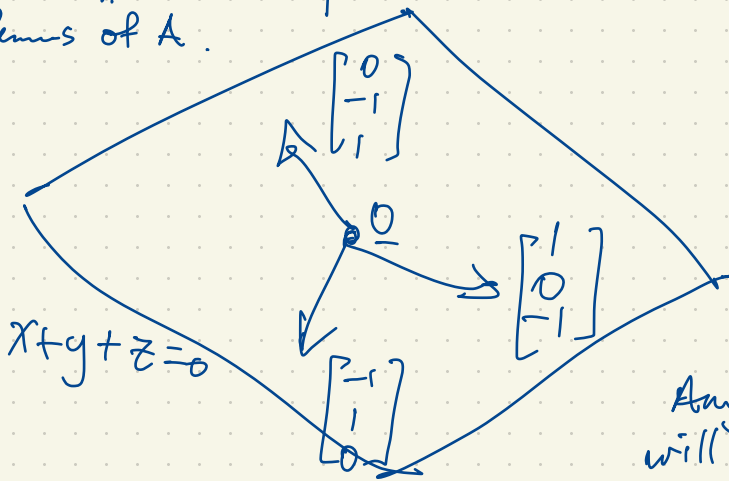
(here \mathbb{R}^3 consists of 3×1 column vectors)

$$T_A(v) = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y-z \\ -x+z \\ x-y \end{bmatrix}$$

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\text{The image of } T_A \text{ is } \left\{ T_A v : v \in \mathbb{R}^3 \right\} = \left\{ \begin{bmatrix} y-z \\ -x+z \\ x-y \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

The image of T_A is the span of the columns of A .



$$x \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + z \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

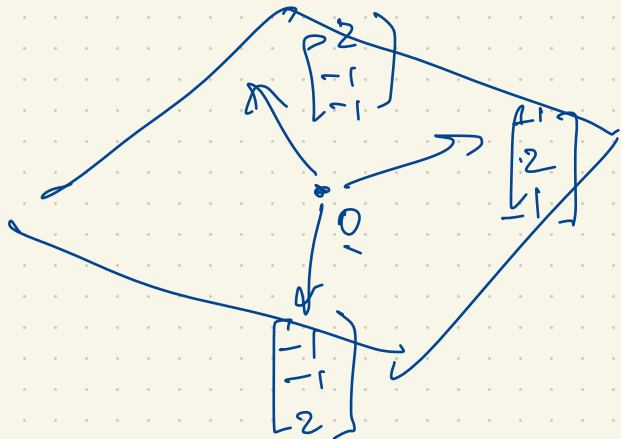
(a linear combination of the columns of A)

T_A is not onto \mathbb{R}^3 . This happens because the columns of A fail to span \mathbb{R}^3 .

Any 3 linearly independent vectors in \mathbb{R}^3 will span all of \mathbb{R}^3 (their span is \mathbb{R}^3).

Another example: $B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ defines a linear transformation $T_B: \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

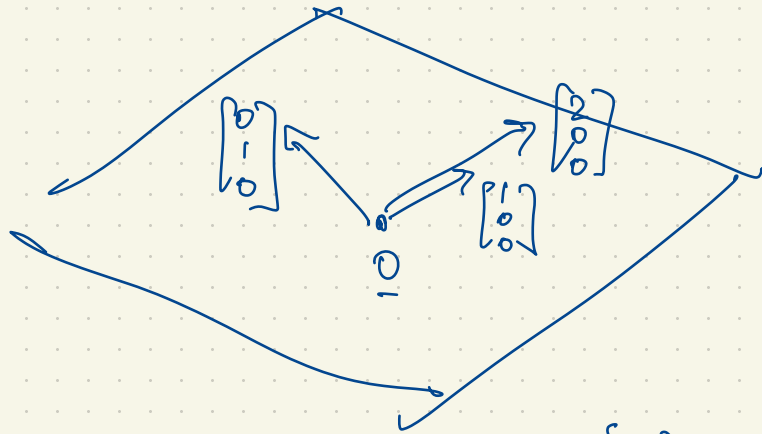
Once again T_B is not onto \mathbb{R}^3 ; its image is the span of the columns of B i.e. the plane $x+y+z=0$ through the origin in \mathbb{R}^3 .



$C = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ has three linearly independent columns spanning \mathbb{R}^3 i.e. the image of T_C is \mathbb{R}^3 i.e. T_C is onto \mathbb{R}^3 .

Check: If $a \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + c \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3a-b-c \\ -a+2b-c \\ -a-b+2c \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has $\left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$ as the span of its columns.
 T_A is not onto.



The span of the rows of A is $\{ [a, 2a, b] : a, b \in \mathbb{R} \}$

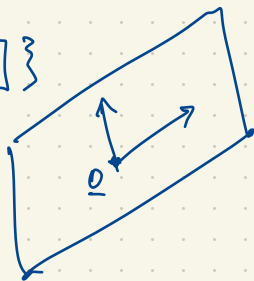
A subspace of \mathbb{R}^n generalizes the notion of $\{0\}$, line through the origin, plane through the origin, etc. up to and including \mathbb{R}^n itself. The dimension of such a subspace is $0, 1, 2, 3, \dots, n$.

Given any set $S \subset \mathbb{R}^n$ (any set of vectors) then $\text{span } S = \{ \text{linear combinations of vectors in } S \}$ is a subspace of \mathbb{R}^n . Another way is to solve any homogeneous linear system in n variables.

The latter case is the same thing as finding the null space of a linear transformation. In particular if A is an $m \times n$ matrix then $\text{Nul } A = \left\{ \underset{\substack{\uparrow \\ \text{in } \mathbb{R}^m}}{v} \in \mathbb{R}^n : Av = \underset{\substack{\uparrow \\ \text{in } \mathbb{R}^m}}{0} \right\}$ is a subspace of \mathbb{R}^n .

Ex. a 2-dimensional subspace of \mathbb{R}^3 (i.e. a plane through the origin) can be described in either of two ways.

$$U = \text{Span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \right\}$$



$$x + 3y - z = 0$$



Alternatively, $U = \text{Nul} \begin{bmatrix} 1 & 3 & -1 \end{bmatrix}$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \right\}$$

$$= \left\{ s \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

Ex. a 1-dimensional subspace of \mathbb{R}^3 (i.e. a line through the origin).

$$U = \text{span} \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right\}$$



$$U = \text{Nul} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

i.e. $\begin{cases} x + y + z = 0 \\ x + 2y + 4z = 0 \end{cases}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

$$U = \text{Nul} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

x, y are basic variables;
 z is a free variable.

$z = t$ where t is arbitrary; solve for y, x

$$y = -3t$$

$$x = 2t$$

$$U = \left\{ \begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

The solutions of $y''+y=0$ form a vector space $\{y : y''+y=0\} = \text{span}\{\sin x, \cos x\}$
 $= \{a \sin x + b \cos x : a, b \in \mathbb{R}\}$

Here $Ty = y''+y$ is a function mapping one function to another. $= \text{Nul } T.$

$$T: \{\text{functions}\} \rightarrow \{\text{functions}\}$$

T is a linear transformation since $T(ay_1 + by_2) = aTy_1 + bTy_2.$

Let $T: V \rightarrow W$ be a linear transformation.

T is one-to-one iff $\text{Nul } T = 0.$

T is onto iff every $w \in W$ has the form $w = Tv$ for some $v \in V.$

T is bijective iff it is both one-to-one and onto. Such functions T have an inverse $T^{-1}.$

T^{-1} must also be linear.

Eg. consider the 2×2 matrix $A = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$ which represents a linear transformation $T_A: \mathbb{R}^2 \rightarrow \mathbb{R}^2.$

Find the inverse matrix $A^{-1}.$

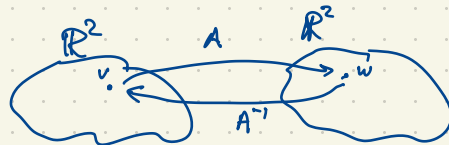
$$A^{-1}(Av) = v$$

$$A(A^{-1}w) = w$$

$$A^{-1}A = I$$

$$AA^{-1} = I$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ identity}$$



Fri. Oct 13 Quiz: Inverses of Matrices

A 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible iff $ad-bc \neq 0$, in which case $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Eg. for $A = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$ we have $3 \cdot 5 - 2 \cdot 8 = -1$, $A^{-1} = \frac{1}{-1} \begin{bmatrix} 5 & -2 \\ -8 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 8 & -3 \end{bmatrix}$.

Check: $AA^{-1} = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^{-1}A = I$.

Eg. $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$. Compute B^{-1} .

General method: To compute A^{-1} , if it exists, write down $\begin{bmatrix} A & | & I_n \end{bmatrix}$ and row reduce leading to $\begin{bmatrix} I_n & | & A^{-1} \end{bmatrix}$

$\begin{matrix} \nearrow n \times n & \nwarrow n \times 2n \\ I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \end{matrix}$

In our case $[B | I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 2 & 8 & -1 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 2 & 8 & -1 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -\frac{5}{2} & 4 & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array} \right]$

$$B^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

Check: $B^{-1}B = \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

$n \times 2n$
If the pivots are not all in the leftmost n columns, we don't get I_n on the left. In this case A is not invertible.

Eg. $A = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$

$$[A | I] = \left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 8 & 5 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ -1 & -1 & -3 & 1 \end{array} \right] \sim \left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 1 & 1 & 3 & -1 \end{array} \right] \sim \left[\begin{array}{cc|cc} 0 & -1 & -8 & 3 \\ 1 & 1 & 3 & -1 \end{array} \right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 3 & -1 \\ 0 & -1 & -8 & 3 \end{array} \right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 1 & 3 & -1 \\ 0 & 1 & 8 & -3 \end{array} \right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 8 & -3 \end{array} \right]$$

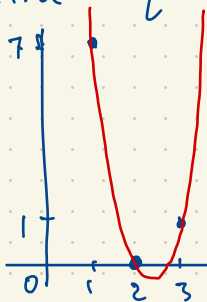
$$A^{-1} = \begin{bmatrix} -9 & 2 \\ 8 & -3 \end{bmatrix}$$

Eg. $A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$ has $3 \cdot 2 - 1 \cdot 6 = 0$ so A is not invertible. What goes wrong in our algorithm?

$$[A | I] = \left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 6 & 2 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array} \right] \sim \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -2 & 1 \end{array} \right] \sim \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -2 & 1 \end{array} \right]$$

The pivots do not appear in the leftmost two columns so we conclude that A is not invertible. The image of T_A is the span of the columns of A , namely $\text{span} \left\{ \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$, not \mathbb{R}^2 . So T_A is not invertible i.e. A is not invertible.

Eg. Find a quadratic polynomial $f(t) = at^2 + bt + c$ having table of values



$$f(1) = c + b + a = 7$$

$$f(2) = c + 2b + 4a = 0$$

$$f(3) = c + 3b + 9a = 1$$

$$\text{So } f(t) = 22 - 19t + 4t^2$$

check: $f(1) = 7, f(2) = 0, f(3) = 1$ ✓

t	f(t)
1	7
2	0
3	1

Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ -\frac{3}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ -19 \\ 4 \end{bmatrix}$$

The solution of a linear system $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$ assuming A is an invertible $n \times n$ matrix. $[A | I] \sim \dots \sim [I | A^{-1}]$

$A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$ is not invertible since the span of its columns is $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ i.e. A has linearly dependent columns. $\begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Alternatively, A has a null vector $\begin{bmatrix} 1 \\ -3 \end{bmatrix} \in \text{Nul } A$ since $A\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = A\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\text{Nul } A = \text{span}\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}$ so A is not one-to-one.

The linear system $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has many solutions.

The linear system $A\mathbf{x} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ has no solutions. since $\begin{bmatrix} 1 \\ 7 \end{bmatrix} \notin \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

In 5th edition, I'm omitting

2.4	Partitioned Matrices
2.5	Matrix Factorizations
2.6	Leontief-Input/Output Model
2.7	Computer graphics

$$U_1 \cap U_2 = \{u : u \in U_1 \text{ and } u \in U_2\}$$

Continue with 2.8: Subspaces of \mathbb{R}^n

A subspace of \mathbb{R}^n is a subset $U \subseteq \mathbb{R}^n$ such that

- (i) $\mathbf{0} \in U$
- (ii) For all $u, v \in U$, $u+v \in U$.
- (iii) For all $u \in U$ and scalar $c \in \mathbb{R}$, $cu \in U$.

Eg. In \mathbb{R}^2 , $\{(x, y) : x, y \geq 0\}$ is not a subspace.

Think of: $\{0\}$, line through the origin, plane through the origin, etc.

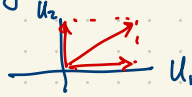
If U_1, U_2 are subspaces of \mathbb{R}^n , is $U_1 \cap U_2$ also a subspace of \mathbb{R}^n ?

- (i) Since $\mathbf{0} \in U_1$ and $\mathbf{0} \in U_2$, $\mathbf{0} \in U_1 \cap U_2$.
- (ii) Let $u, v \in U_1 \cap U_2$. Then $u+v \in U_1$ and $u+v \in U_2 \Rightarrow u+v \in U_1 \cap U_2$.
- (iii) Let c be a scalar and $u \in U_1 \cap U_2$. Then $cu \in U_1$ and $cu \in U_2$ so $cu \in U_1 \cap U_2$.

So yes, the intersection of two subspaces is a subspace.

$U_1 \cup U_2 = \{u : u \in U_1 \text{ or } u \in U_2\}$ i.e. u is in at least one of U_1 or U_2 , possibly both.

If U_1 and U_2 are subspaces of \mathbb{R}^n , must $U_1 \cup U_2$ also be a subspace? No.



eg. $U_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} =$ the x -axis in \mathbb{R}^2

$U_2 = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} =$ the y -axis in \mathbb{R}^2

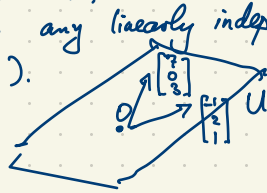
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin U_1 \cup U_2.$$

↑ in U_1, U_2 ↑ in U_1, U_2

Alternatively, a subspace is a nonempty subset $U \subseteq \mathbb{R}^n$ such that linear combinations of vectors in U is still in U i.e. $\text{span } U = U$.

If U is a subspace of \mathbb{R}^n ($U \leq V$) then a basis for U is any linearly independent set of vectors spanning U .

eg. in \mathbb{R}^3 , let U be the plane $3x + 5y - 7z = 0$ (through the origin).



Another basis for U is $\left\{ \begin{bmatrix} 0 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} \right\}$

The list of vectors $\left\{ \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for U . These two vectors are linearly independent by inspection. Moreover $\text{span}\left\{ \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\} = U$ (this is not quite obvious but we will soon see why it's true).

The dimension of U is 2 because we have a basis consisting of 2 vectors.

Syllabus

HW	10
Quizzes	20
Test 1	20
Test 2	20
Exam	30
	<hr/>
	100

HW	30
Quizzes	20
Test	20
Exam	30
	<hr/>
	100

I'll correct this online and email everyone with this correction.

How do we find a basis for a subspace of \mathbb{R}^n ?

Eg. If A is an $m \times n$ matrix, $\text{Row } A = \text{span}(\text{rows of } A) \subseteq \mathbb{R}^n$ (really $1 \times n$ vectors)

$\text{Col } A = \text{span}(\text{columns of } A) \subseteq \mathbb{R}^m$ (really $m \times 1$ vectors)

(the row space and column space of A).

Take e.g. $A = \begin{bmatrix} 0 & 1 & -1 & 0 & 3 & 6 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ in reduced row echelon form.

Row A has basis $(0, 1, -1, 0, 3, 6)$, $(0, 0, 0, 1, -5, 2)$ so Row A is 2-dimensional: $\dim(\text{Row } A) = 2$.

The dimension of $U \subseteq \mathbb{R}^n$ is the number of vectors in a basis for U .

Col A has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Col $A = \text{span}(\text{columns of } A)$

$$= \left\{ c_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_5 \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix} + c_6 \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} : c_1, c_2, \dots, c_6 \text{ any scalars} \right\}$$

$$= \left\{ c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} : c_2, c_4 \text{ scalars} \right\} = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\} \text{ (the } xy\text{-plane)}$$

$\dim \text{Col } A = 2$.

Although row vectors have length 6 and column vectors have length 3, the row space and column space have the same dimension. (equal to the number of pivots).

What if A is not in reduced row echelon form?

Eg. $B = \begin{bmatrix} 0 & 2 & -2 & 1 & 1 & 12 \\ 0 & 1 & -1 & 3 & -12 & 2 \\ 0 & 1 & -1 & -2 & 13 & 2 \end{bmatrix}$

Row $B \subseteq \mathbb{R}^6$
Col $B \subseteq \mathbb{R}^3$

$$B \sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 2 & -2 & 1 & 1 & 14 \\ 0 & 1 & -1 & -2 & 13 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 0 & 0 & -5 & 25 & -10 \\ 0 & 1 & -1 & -2 & 13 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 1 & -1 & -2 & 13 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & -5 & 25 & -10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 1 & -5 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 0 & 3 & 6 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A$$

Row $B =$ Row A has basis $(0, 1, -1, 0, 3, 6), (0, 0, 0, 1, -5, 2)$

Col $B \neq$ Col A but Col B has basis $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

In general, the pivot columns of A (= reduced row echelon form of B) tell us which columns of B give a basis for col B .

e.g. $\begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} -12 \\ 13 \\ 13 \end{bmatrix} = (3) \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + (-5) \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

\dim Row $B = \dim$ Col $B =$ no. of pivots = rank B
 \dim Nul $B = (\text{no. of columns of } B) - (\text{no. of pivots})$

The rank of a matrix is the dimension of its row and column space. The nullity of a matrix is the dimension of its null space.

Fact: Although Row B and Col B are very different (one is a set of 1×6 row vectors; the other is a set of 3×1 column vectors) they have the same dimension; in each case the dimension is the number of pivots of A , the reduced row-echelon form of B .

Another important subspace related to B is its null space Nul $B =$ Nul A which has basis

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \begin{bmatrix} 0 & 1 & -1 & 0 & 3 & 6 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

basic variables x_2, x_4
free variables x_1, x_3, x_5, x_6
Choose parameters r, s, t, u

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} r \\ s - 3r - 6s \\ s \\ 5t - 2u \\ t \\ u \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -3 \\ 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ -6 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Another way to get a basis for the column space of B is to transpose the matrix B to obtain its transpose

$$B^T = \begin{bmatrix} 0 & 2 & -2 & 1 & 1 & 14 \\ 2 & 1 & -1 & 3 & -12 & 12 \\ -2 & -1 & -1 & 3 & -12 & 12 \\ 1 & 3 & -2 & 1 & -12 & 12 \\ 1 & 3 & -2 & 1 & -12 & 12 \\ 14 & 12 & 12 & 1 & -12 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank } B^T = 2$$

A basis for the row space of B^T is $(1, 0, 1)$, $(0, 1, -1)$;

a basis for the column space of B^T is $\begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \\ 1 \\ 14 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 3 \\ -12 \\ 12 \\ 12 \end{bmatrix}$.

So: a basis for the column space of B is $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$;

and a basis for the row space of B is $(0, 2, -2, 1, 1, 14)$, $(0, 1, -1, 3, -12, 12)$
(the first two rows of B).

$$B = \begin{bmatrix} 0 & 2 & -2 & 1 & 1 & 14 \\ 2 & 1 & -1 & 3 & -12 & 12 \\ -2 & -1 & -1 & 3 & -12 & 12 \\ 1 & 3 & -2 & 1 & -12 & 12 \\ 1 & 3 & -2 & 1 & -12 & 12 \\ 14 & 12 & 12 & 1 & -12 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = (2) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -12 \\ 12 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (-12) \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

If $A = \begin{bmatrix} 5 & 3 \\ 7 & -1 \end{bmatrix}$ then $A \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 7 & -1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 5x+3z \\ 7x-z \end{bmatrix}$ and $A \begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} 5y+3w \\ 7y-w \end{bmatrix}$ so

$$A \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 5x+3z & 5y+3w \\ 7x-z & 7y-w \end{bmatrix}$$

$$A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 7 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x+3z & y+3w \\ z & w \end{bmatrix}$$

↑ This matrix is an elementary matrix; it corresponds to an elementary row operation of adding $3 \times$ row 2 to row 1.