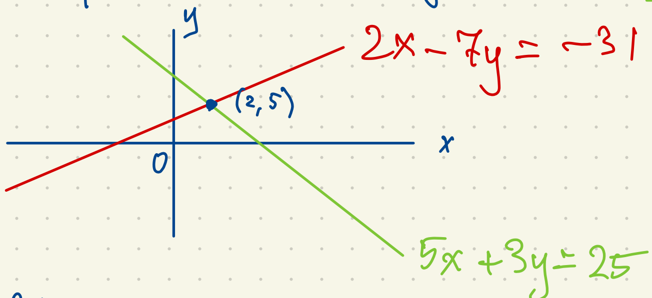
Linear Algebra

Book 1

Example: Find all (x,y) such that $5x+3y=25$ and $2x-7y=-31$.



Solution: $(x,y) = (2,5)$ is the unique solution.

We are asking for the simultaneous solution of a system of two equations in two unknowns x and y.

$$\begin{cases} 5x + 3y = 25 & (1) \\ 2x - 7y = -31 & (2) \end{cases}$$

$2\times(1) - 5\times(2) = (3)$

$2\times 3 - 5(-7) = 6+35 = 41$

$2\times 25 - 5\times(-31) = 50+155 = 205$

$41y = 205$

$(4) = (3) \div 41$

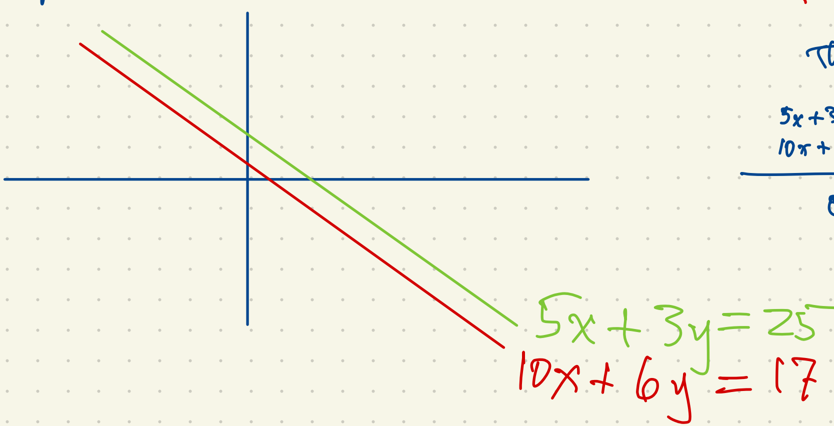
$y = 5$

$5x + 15 = 25$

$5x = 10$

$x = 2$

Example: Find all (x,y) such that $5x+3y=25$ and $10x+6y=17$.



This system is inconsistent: it has no solution.

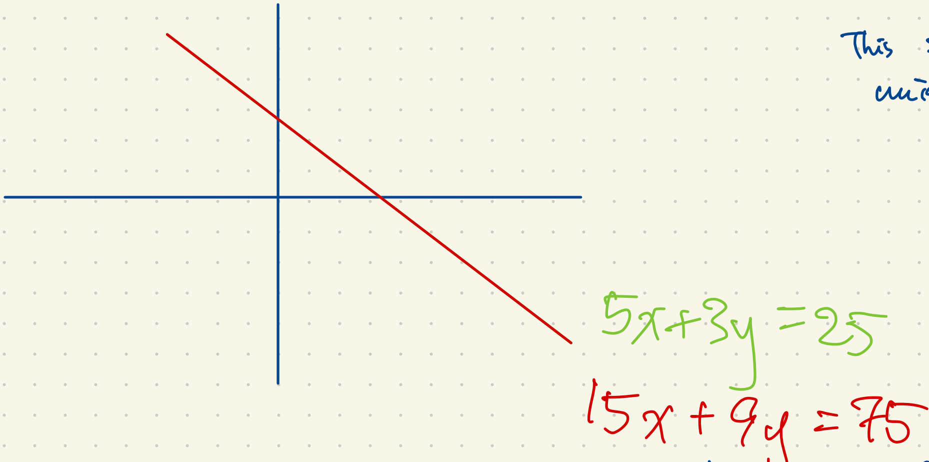
$5x+3y=25$ (1)

$10x+6y=17$ (2)

$0 = 33$ $\quad 2\times(1) - (2)$

This is inconsistent.

Example : Find all (x,y) such that $5x+3y=25$ and $15x+9y=75$.



This system is consistent but the solution is not unique: there are infinitely many solutions.

$$\begin{array}{rl} 5x+3y=25 & (1) \\ 15x+9y=75 & (2) \\ \hline 0=0 & (3)=3\times(1)-(2) \end{array}$$

A system of m linear equations in n unknowns has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

(a_{ij}, b_i constants for $i \in \{1,\cdots,m\}$, $j = \{1,2,\cdots,n\}$; $x_1,\cdots,x_n$ variables representing unknowns).

Typically, when $m=n$ we can expect a unique solution;
$m>n$ — — — : no solution (inconsistent system);
$m<n$ — — : more than one solution.

Example with $m=n=3$: a system of 3 linear equations in 3 unknowns.

Kim buys a bag of 26 items weighing 226 oz. costing \$34. The items included

cans of tuna (\$1 each, 5 oz each)

apples (\$1 each, 8 oz each)

loaves of bread (\$3 each, 20 oz each)

How many of each item did Kim buy? (say x cans of tuna, y apples, z loaves of bread)

$$\begin{aligned} x + y + z &= 26 && (1)\\ 5x + 8y + 20z &= 226 && (2)\\ x + y + 3z &= 34 && (3) \end{aligned}$$

$$\begin{aligned} 2z &= 8 && (3)-(1) = (4)\\ z &= 4 && (5)\\ x + y &= 22 && (6) = (8)-(5)\\ 5x + 8y &= 146 && (7)\\ 3y &= 36 && (7) - 5\times(6) = (8)\\ y &= 12 && (9) = (8)\div 3\\ x &= 10 && (10) = (6)-(9). \end{aligned}$$

$146 - 5\times 22 = 146 - 110 = 36$

The unique solution of this system is $(x,y,z) = (10,12,4)$. (Kim bought 10 cans of tuna, 12 apples, and 4 loaves of bread.)

Check: that all three equations are satisfied.

Matrix formulation of linear systems

$$\begin{aligned} x + y + z &= 26 \\ 5x + 8y + 20z &= 226 \\ x + y + 3z &= 34 \end{aligned} \longrightarrow \begin{array}{cccc} x & y & z & \text{total} \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 1 & 1 & 3 & 34 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 1 & 1 & 3 & 34 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 0 & 0 & 2 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 0 & 3 & 15 & 96 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 0 & 1 & 5 & 32 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

subtract row 1 from row 3; divide row 3 by 2; subtract 5 times row 1 from row 2; divide row 2 by 3

$$226 - 5 \times 26 = 226 - 130 = 96$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 14 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{array}\right] \quad \text{i.e.} \quad \begin{aligned} x &= 10 \\ y &= 12 \\ z &= 4 \end{aligned}$$

subtract 5 times row 3 from row 2; subtract row 2 from row 1; subtract row 3 from row 1

Example: Find all (x,y) such that $5x+3y=25$ and $2x-7y=-31$.

$$\begin{array}{cc} x & y \end{array} \left[\begin{array}{cc|c} 5 & 3 & 25 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3/5 & 5 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{3}{5} & 5 \\ 0 & -\frac{41}{5} & -41 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{3}{5} & 5 \\ 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$$

divide row 1 by 5; subtract 2 times row 1 from row 2; multiply row 2 by $-\frac{5}{41}$; subtract $\frac{3}{5}$ times row 2 from row 1

$$-7 - \frac{6}{5} = -\frac{35}{5} - \frac{6}{5} = -\frac{41}{5}$$

$$-31 - 10 = -41$$

Solution: $(x, y) = (2,5)$.

Alternatively:

$$\left[\begin{array}{cc|c} 5 & 3 & 25 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -7 & -31 \\ 5 & 3 & 25 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{7}{2} & -\frac{31}{2} \\ 5 & 3 & 25 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$$

interchange rows 1 and 2

Even better: $\left[\begin{array}{cc|c} 5 & 3 & 25 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 17 & 87 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 17 & 87 \\ 0 & -41 & -205 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 17 & 87 \\ 0 & 1 & 5 \end{array}\right]$

subtract 2 times row 2 from row 1 — subtract 2 times row 1 from row 2 — divide row 2 by -41

$-31 - 2\times 87$
$= -31 - 174$
$= -205$

$\sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$

subtract 17 times row 2 from row 1

Solution: $(x,y) = (2,5)$.

Check! $5\times 2 + 3\times 5 = 25$
$2\times 2 - 7\times 5 = -31$

Elementary row operations:
(i) add a multiple of one row to another
(ii) multiply a row by a nonzero constant
(iii) interchange two rows

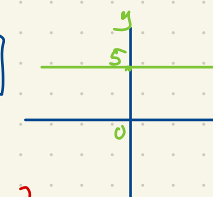
$A \sim B$ means that A, B are linear systems having the same solutions.
We use Gaussian elimination to reduce $A_1 \sim A_2 \sim \cdots \sim A_m$ where A_1 represents the linear system and A_m represents an equivalent linear system (ie. having the same solutions) but A_m is simpler than A_1. Each step $A_i \sim A_{i+1}$ is obtained by one elementary row operation.
Why just one operation at a time?

$$\left.\begin{array}{l} 5x + 3y = 25 \\ 2x - 7y = -31 \end{array}\right\} \quad \left[\begin{array}{cc|c} 5 & 3 & 25 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{3}{5} & 5 \\ 1 & -\frac{7}{2} & -\frac{31}{2} \end{array}\right] \not\sim \left[\begin{array}{cc|c} 0 & \frac{41}{10} & \frac{41}{2} \\ 0 & -\frac{41}{10} & -\frac{41}{2} \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 5 \\ 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

divide row 1 by 5
divide row 2 by 2

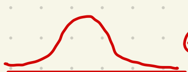
subtract row 2 from row 1
subtract row 1 from row 2

ie. $y = 5$
$0 = 0$



unique solution $(2,5)$

infinitely many solutions

Gauss  Gaussian distribution

$\left[\begin{array}{cc|c} 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$, $\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 0 & 4 \end{array}\right]$, $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$ are examples of matrices in <u>reduced row echelon form</u>: they cannot be simplified any further by elementary row operations.

$\left[\begin{array}{cc|c} 1 & 17 & 87 \\ 0 & 1 & 5 \end{array}\right]$ is almost reduced; it is in <u>row echelon form</u>.

For a linear system whose matrix is in row echelon form, we can solve for the unknowns $x_1, x_2, \cdots, x_n$; we solve for x_n, then x_{n-1}, then x_{n-2}, $\cdots$, x_1 by <u>back-substitution</u>.

eg. $\begin{array}{c} \begin{matrix} x & y & z & \end{matrix} \\ \left[\begin{array}{ccc|c} 5 & 3 & 7 & 3 \\ 0 & 2 & 11 & 4 \\ 0 & 0 & 6 & 8 \end{array}\right] \end{array}$ is in row echelon form.

Every linear system has a unique reduced row echelon form.

In any $m \times n$ matrix, a <u>pivot</u> is the first nonzero entry in its row.
(Pivots are highlighted above.)

In order for a matrix to be in <u>row echelon form</u>, we must have
- pivots in any row must occur to the right of pivots in any previous rows;
- any zero rows occur at the bottom.

Assuming a matrix is already in row echelon form, then to be in <u>reduced row echelon form</u>, we must have
- every pivot entry must be a 1
- every column having a pivot has only one nonzero entry.

Example: Solve the following linear system of 3 equations in 5 unknowns:

$$\begin{cases} x_1 + 4x_2 - x_3 + 2x_4 + 3x_5 = 6 \\ 2x_1 + 8x_2 - x_3 + 7x_4 + 4x_5 = 19 \\ -x_1 - 4x_4 + 4x_3 + 8x_4 - 4x_5 = 26 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 2 & 8 & -1 & 7 & 4 & 19 \\ -1 & -4 & 4 & 8 & -4 & 26 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ -1 & -4 & 4 & 8 & -4 & 26 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 3 & 10 & -1 & 32 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right]$$

This matrix is in row echelon form. This can be used to solve the linear system by back-substitution.

$$x_4 + 5x_5 = 11$$

$x_5 = t$ is a free parameter.

$$x_4 = 11 - 5t$$

$$x_3 + 3x_4 - 2x_5 = 7$$

$$x_3 = 7 - 3x_4 + 2x_5 = 7 - 3(11-5t) + 2t = -26 + 17t$$

$$x_1 + 4x_2 - x_3 + 2x_4 + 3x_5 = 6$$

$x_2 = s$ is another free parameter

$$x_1 = 6 - 4x_2 + x_3 - 2x_4 - 3x_5 = 6 - 4s + (-26+17t) - 2(11-5t) - 3t$$

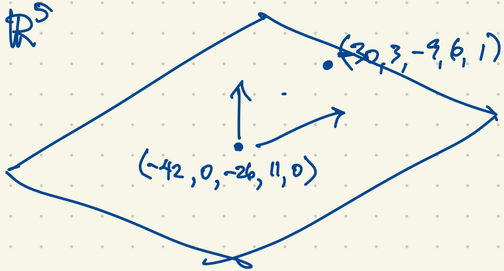
$$= -42 - 4s + 24t$$

Solutions: $(x_1, x_2, x_3, x_4, x_5) = (-42 - 4s + 24t,\ s,\ -26 + 17t,\ 11 - 5t,\ t)$ where s, t are arbitrary.

Geometrically, the set of solutions forms a plane (2-dimensional surface) in $\mathbb{R}^5$.

two parameters s, t are coordinates for the plane

$\mathbb{R}^5$



Solution set inside $\mathbb{R}^5$.

The (solution) point corresponding to $(s,t) = (3,1)$ is $(30, 3, -9, 6, 1)$ is another solution.

Our system is consistent but the solution is not unique.

$$\left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 5 & 1 & 13 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 0 & -24 & -42 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 0 & -24 & -42 \\ 0 & 0 & 1 & 0 & -17 & -26 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right]$$

(row echelon form) (reduced row echelon form)

To solve a linear system in reduced row echelon form, introduce parameters for the free variables (the variables whose columns do not contain a pivot).

In the example above, x_2 and x_5 are the free variables. Introduce s, t. $x_2 = s$, $x_5 = t$ can be chosen freely. Solve for the variables x_1, x_3, x_4 using the equations appearing in the reduced row echelon form:

$$\left.\begin{aligned} x_1 + 4s - 24t &= -42 \\ x_3 - 17t &= -26 \\ x_4 + 5t &= 11 \end{aligned}\right\} \Rightarrow (x_1, x_2, x_3, x_4, x_5) = (-42 - 4s + 24t,\ s,\ -26 + 17t,\ 11 - 5t,\ t)$$

where s, t are arbitrary.

(This is the parametric solution in terms of the parameters s, t. The system is consistent, having infinitely many solutions.)

As long as the rightmost column has no pivot, the system is consistent.

The general solution can be written as

$$\begin{aligned} (x_1, x_2, x_3, x_4, x_5) &= (-42 - 4s + 24t,\ s,\ -26 + 17t,\ 11 - 5t,\ t) \\ &= (-42, 0, -26, 11, 0) + s(-4, 1, 0, 0, 0) + t(24, 0, 17, -5, 1) \end{aligned}$$

$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$ (vector addition)

$\underset{\text{scalar}}{c}\,\underbrace{(a_1, a_2, \dots, a_n)}_{\text{vector}} = (ca_1, ca_2, \dots, ca_n)$ (scalar multiplication)

Algebraic operations for matrices

If $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix}$ then $BA = \underbrace{\begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \end{bmatrix}}_{2\times 3} = \underbrace{\begin{bmatrix} 13 & 11 & 41 \\ -1 & -27 & 23 \end{bmatrix}}_{2\times 3}$

Here AB is undefined.

An $m\times n$ matrix has the form

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

$a_{i,j}$ is the (i,j)-entry of the matrix A.

$i \in \{1, 2, \dots, m\}$

$j \in \{1, 2, \dots, n\}$.

Often $a_{i,j}$ is written a_{ij}. (unless this results in confusing)

If A is $m\times n$ and B is $n\times r$ then $\underbrace{A}_{m\times n}\underbrace{B}_{n\times r}$ is $m\times r$.

We can't multiply two matrices unless the number of columns in the first matrix equals the number of rows in the second matrix.

eg. $\underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \end{bmatrix}}_{2\times 3}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}}_{3\times 2} = \begin{bmatrix} 12 & 8 \\ 23 & 4 \end{bmatrix}$ whereas $\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}}_{3\times 2}\underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \end{bmatrix}}_{2\times 3} = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \\ 5 & -1 & 21 \end{bmatrix}$

If $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$ then $A^2 = \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}}_{A} = \begin{bmatrix} 7 & 3 \\ 1 & 4 \end{bmatrix}$, $A^3 = A^2 A = \begin{bmatrix} 7 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 17 & 18 \\ 6 & -1 \end{bmatrix}$

Recall: the linear system $\left.\begin{aligned} 5x + 3y &= 25 \\ 2x - 7y &= -31 \end{aligned}\right\}$ has a unique solution $(x,y) = (2,5)$.

One way to solve this: Write the linear system as $\underline{A}\underline{v} = \underline{b}$ where A is a 2×2 matrix, $v = \begin{bmatrix} x \\ y \end{bmatrix}$ is a 2×1 matrix (i.e. column vector of length 2) and $b = \begin{bmatrix} 25 \\ -31 \end{bmatrix}$ is a 2×1 matrix of constants.

Here $A = \begin{bmatrix} 5 & 3 \\ 2 & -7 \end{bmatrix}$.

$Av = b$ says $\underbrace{\begin{bmatrix} 5 & 3 \\ 2 & -7 \end{bmatrix}}_{2\times \cancel{2}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\cancel{2}\times 1} = \underbrace{\begin{bmatrix} 25 \\ -31 \end{bmatrix}}_{2\times 1}$ i.e. $\begin{bmatrix} 5x+3y \\ 2x-7y \end{bmatrix} = \begin{bmatrix} 25 \\ -31 \end{bmatrix}$

Compare: To solve $3x = 5$, multiply both sides by $3^{-1} = \frac{1}{3}$ on the left: $\cancel{3}^{-1}\cancel{3}x = 3^{-1}5$ i.e. $x = \frac{5}{3}$.

To solve $Av = b$, multiply both sides on the left by $A^{-1} = \frac{1}{41}\begin{bmatrix} 7 & 3 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} \frac{7}{41} & \frac{3}{41} \\ \frac{2}{41} & -\frac{5}{41} \end{bmatrix}$

$$Av = b$$
$$A^{-1}Av = A^{-1}b$$

$$\frac{1}{41}\begin{bmatrix} 7 & 3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 2 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{41}\begin{bmatrix} 7 & 3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} 25 \\ -31 \end{bmatrix}$$

$$\frac{1}{41}\begin{bmatrix} 41 & 0 \\ 0 & 41 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{41}\begin{bmatrix} 82 \\ 205 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\substack{I_2 \text{ is the} \\ 2\times 2 \text{ identity matrix}}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$
($n\times n$ identity matrix)

If $\underbrace{\underbrace{(\underset{2\times 7}{A}\underset{7\times 3}{B})}_{2\times 3}\underset{3\times 5}{C}}_{2\times 5} = \underbrace{\underset{2\times 7}{A}\underset{7\times 5}{(BC)}}_{2\times 5}$ by associativity, you can do the first way since that is faster.

We say A and B commute if $AB = BA$.

Which 2×2 matrices commute with $A = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$? Answer by solving the appropriate linear system of 4 equations in 4 unknowns. Let $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. In order for $AB = BA$ we require

$$\begin{bmatrix} 3x+z & 3y+w \\ 4z & 4w \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3x & x+4y \\ 3z & z+4w \end{bmatrix}$$

i.e. $\begin{cases} 3x+z = 3x \\ 3y+w = x+4y \\ 4z = 3z \\ 4w = z+4w \end{cases}$

$$\begin{array}{cccc} x & y & z & w \end{array}$$
$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

x, z are basic variables (in the pivot columns)
y, w are free variables (non-pivot columns)

Introduce s, t as parameters. $y = s$, $w = t$
and solve for x, z: $x = -s+t$, $z = 0$ so $B = \begin{bmatrix} -s+t & s \\ 0 & t \end{bmatrix} = s\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(a linear combination of $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

Check: If $s=1, t=0$ then $B = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$ and $BA = AB$.

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}$$

$IA = A = AI$

For Friday's Quiz: Representation of linear systems in matrix form. $Ax = b$
Particular and general solutions
Null vectors
Homogeneous linear systems $Ax = 0$

$Ax = b \xrightarrow{\text{homogenize}} Ax = 0$

$\underset{m\times n}{\mathrm{Nul}(A)} = \{ \underset{n\times 1}{x} \in \mathbb{R}^n : \underset{m\times n}{A}\,\underset{n\times 1}{x} = \underset{m\times 1}{0} \in \mathbb{R}^m \}$ is the null space of A. Its vectors are called null vectors.

$$\begin{cases} x_1 + 4x_2 - x_3 + 2x_4 + 3x_5 = 6 \\ 2x_1 + 8x_2 - x_3 + 7x_4 + 4x_5 = 19 \\ -x_1 - 4x_4 + 4x_3 + 8x_4 - 4x_5 = 26 \end{cases}$$

$$\underbrace{\begin{bmatrix} 1 & 4 & -1 & 2 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}}_{3\times 5} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{5\times 1} = \underbrace{\begin{bmatrix} 6 \\ 7 \\ 11 \end{bmatrix}}_{3\times 1}$$

Every linear system has the form $\underbrace{A}_{m\times n}\underbrace{x}_{n\times 1} = \underbrace{b}_{m\times 1}$ (for m linear equations in n unknowns).

$$A = \begin{bmatrix} 1 & 4 & -1 & 2 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 7 \\ 11 \end{bmatrix}$$

Some prefer to write $\vec{x}$ or $\overrightarrow{x}$ instead of x. or even $\mathbf{x}$ (bold face) or $\underline{x}$. For us, context is used to determine whether we are talking about a matrix, a vector, a scalar, a set, a linear transformation, a vector space, etc.

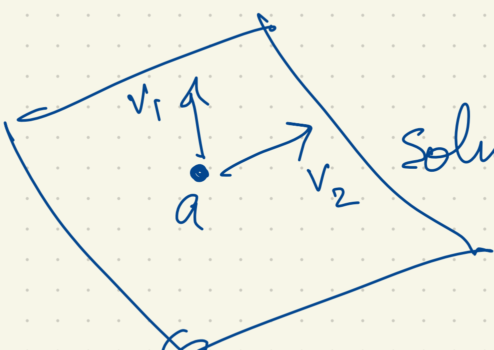
Some linear systems are inconsistent (meaning that they have no solutions). If a linear system $Ax=b$ is consistent then its solutions have the form $x = a + c_1v_1 + c_2v_2 + \cdots + c_kv_k$ for some particular solution $x=a$; $c_1, \dots, c_k$ scalars (constants) and $v_1, \dots, v_k$ are independent solutions of $Ax=0$.

The solutions of the example $Ax=b$ above have the form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \boxed{\underbrace{\begin{bmatrix} -42 \\ 0 \\ -26 \\ 11 \\ 0 \end{bmatrix}}_{a} + s\underbrace{\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + t\underbrace{\begin{bmatrix} 24 \\ 0 \\ 17 \\ -5 \\ 1 \end{bmatrix}}_{v_2} = \underbrace{\begin{bmatrix} -42-4s+24t \\ s \\ -26+17t \\ 11-5t \\ t \end{bmatrix}}_{5\times 1}}$$

a particular solution

The general solution

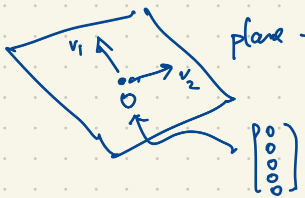


solution set (i.e. the set of <u>all</u> solutions)

$A = \begin{bmatrix} 1 & 4 & -1 & 2 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$ gives rise to a homogeneous linear system $A x = \underbrace{0}_{3\times 1}$ i.e. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -42 \\ \vdots \\ \vdots \end{bmatrix}$

i.e. $\left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \end{array}\right]$ has solutions $x = s v_1 + t v_2$ (no "a" here)



plane through the origin in $\mathbb{R}^5$
(solutions of $Ax = 0$)

Systems of the form $Ax = 0$ are called <u>homogeneous</u> meaning if u and v are solutions then so is $su + tv$.
(i.e. $Au = 0$ and $Av = 0$ then $A(su + tv) = 0$
$sAu + tAv = 0$

A homogeneous system $Ax = 0$ is always consistent, since the zero vector $x = 0$ is a solution.

$\mathrm{Nul}(A)$ = null space of A = $\{$ all solutions of the homogeneous system $\underbrace{A}_{m\times n}\underbrace{x}_{n\times 1} = \underbrace{0}_{m\times 1}\} \subseteq \mathbb{R}^n$

$x \in \mathbb{R}^n$ $\quad 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^m$

$\mathrm{Nul}(A)$ might be: the origin in $\mathbb{R}^n$,
or maybe a line through the origin in $\mathbb{R}^n$,
.. - - - - . plane . - - . - -

Checking answers: If we reduce $A \sim A'$, how can we check our work? (A' could be a row echelon form for A, or maybe a reduced row echelon form for A). A and A' have the same null vectors.

eg. $A = \begin{bmatrix} 1 & 4 & -1 & 2 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \sim A' = \begin{bmatrix} 1 & 4 & 0 & 0 & -24 \\ 0 & 0 & 1 & 0 & -17 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$

A' has null vectors $\begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 24 \\ 0 \\ 17 \\ -5 \\ 1 \end{bmatrix}$. Check that they are also null vectors for A.

$$\begin{bmatrix} 1 & 4 & 0 & 0 & -24 \\ 0 & 0 & 1 & 0 & -17 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 24 \\ 0 \\ 17 \\ -5 \\ 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Every null vector for B is also a null vector for AB.

If $Bx = 0$ then $ABx = A0 = 0$

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} -1/3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 5 \\ 0 & 1 & 2 \\ 9 & 8 & 19 \end{bmatrix} \begin{bmatrix} -1/3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Last week's quiz: $\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix}}_{B} = \begin{bmatrix} 3 & 2 & 5 \\ 0 & 1 & 2 \\ 9 & 8 & 19 \end{bmatrix}$

$\underbrace{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}}_{A} = \begin{bmatrix} 5 & 6 \\ 12 & 18 \end{bmatrix}$

If C is an $n \times n$ matrix, the <u>trace</u> of C is the sum of the n entries on the main diagonal of C. (denoted $\operatorname{tr} C$)

If A is $m \times n$ and B is $n \times m$, then $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. = trace

Eg. Solve $y'' + y = x^2$.

A particular solution is $y = x^2 - 2$. Check: $y'' + y = 2 + (x^2 - 2) = x^2$.

The general solution is $y = x^2 - 2 + a\sin x + b\cos x$ where $a, b \in \mathbb{R}$ are arbitrary.

The homogenized equation $y'' + y = 0$ is homogeneous (so if y_1 and y_2 are two solutions, then a linear combination $ay_1 + by_2$ is also a solution). The general solution of $y'' + y = 0$ is $y = a\sin x + b\cos x$.

What about $\sin(10° + x) = (\cos 10°)\sin x + (\sin 10°)\cos x$. (Every solution of $y'' + y = 0$ is a linear combination of $\sin x$ and $\cos x$.)

A linear combination of vectors $v_1, \dots, v_k$ is a vector of the form $c_1v_1 + c_2v_2 + \dots + c_kv_k$ where $c_1, \dots, c_k$ are scalars. Moreover, $\sin x$ and $\cos x$ are linearly independent. We cannot express the general solution of $y'' + y = 0$ as a linear combination of fewer than two basic solutions.

It is correct to say that the general solution of $y'' + y = 0$ is a linear combination of $\sin x$, $\cos x$ and $\sin(10° + x)$, i.e. every solution has the form $y = c_1\sin x + c_2\cos x + c_3\sin(10° + x)$. However this is the same as

$\sin(10° + x) = (\cos 10°)\sin x + (\sin 10°)\cos x$

$y = (c_1 + c_3\cos 10°)\sin x + (c_2 + c_3\sin 10°)\cos x$.

It's also correct to say: every solution is a linear combination of $\sin x$ and $\sin(10° + x)$.

e.g. $\cos x = \underbrace{\left(-\frac{\cos 10°}{\sin 10°}\right)}_{(-\cot 10°)}\sin x + \underbrace{\left(-\frac{1}{\sin 10°}\right)}_{(-\csc 10°)}\sin(10° + x)$

In fact any two solutions of $y'' + y = 0$ can be used to generate all the other solutions by taking linear combinations, as long as neither of your two particular solutions is a scalar multiple of the other.

A list of vectors $v_1, \dots, v_k$ is linearly independent if none of them is a linear combination of the others.

Alternatively, the only way to have $c_1v_1 + c_2v_2 + \dots + c_kv_k = 0$ is if $c_1 = \dots = c_k = 0$.

Eg. $\sin x$, $\cos x$, $\sin(10^\circ + x)$ are linearly dependent (<u>not</u> linearly independent) since $\sin(10^\circ + x)$ is a linear combination of $\sin x$ and $\cos x$.