Fibonacci Numbers

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

$F_0 = 0$
$F_1 = 1$
$F_2 = 1$
$F_3 = 2$
$F_4 = 3$ etc.
$F_5 = 8$

Recursive formula $F_n = \begin{cases} 0, & \text{if } n=0 \\ 1, & \text{if } n=1; \\ F_{n-1}+F_{n-2}, & \text{if } n \geq 2. \end{cases}$

Consider $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}5\\3\end{bmatrix}, \ldots$
$\quad = v_0, \; = v_1, \; = v_2, \; = v_3, \; = v_4$

So $v_n = \begin{bmatrix}F_{n+1}\\F_n\end{bmatrix}$ so $A = \begin{bmatrix}1&1\\1&0\end{bmatrix}$ defines a map $v_n \longmapsto Av_n = v_{n+1}$ i.e. $A\begin{bmatrix}F_{n+1}\\F_n\end{bmatrix} = \begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}F_{n+1}\\F_n\end{bmatrix} = \begin{bmatrix}F_{n+2}\\F_{n+1}\end{bmatrix} = v_{n+1}$.

Starting with $v_0 = \begin{bmatrix}1\\0\end{bmatrix}$, we get $v_1 = Av_0$, $v_2 = Av_1 = A^2 v_0$, $\cdots$, $v_n = \begin{bmatrix}F_{n+1}\\F_n\end{bmatrix} = A^n\begin{bmatrix}1\\0\end{bmatrix}$ = first column of A^n.

$A^2 = \begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix} = \begin{bmatrix}2&1\\1&1\end{bmatrix}$, $A^3 = \begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix} = \begin{bmatrix}3&2\\2&1\end{bmatrix}$, $A^4 = \begin{bmatrix}3&2\\2&1\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix} = \begin{bmatrix}5&3\\3&2\end{bmatrix}$, ...

To find an explicit formula for A^n (and thereby F_n), diagonalize A.

Characteristic polynomial of A:

$\det(A - xI) = \det\left(\begin{bmatrix}1&1\\1&0\end{bmatrix} - \begin{bmatrix}x&0\\0&x\end{bmatrix}\right) = \begin{vmatrix}1-x&1\\1&-x\end{vmatrix} = (1-x)(-x) - 1 = x^2 - x - 1 = (x-\alpha)(x-\beta)$ where $\underbrace{\alpha = \frac{1+\sqrt5}{2}}_{\text{golden ratio}}$, $\underbrace{\beta = \frac{1-\sqrt5}{2}}_{-0.618\ldots}$

golden ratio $\approx 1.618\ldots$

Eigenvector for α: solution of $Av = \alpha v$ i.e. $(A - \alpha I)v = 0$

$\begin{bmatrix}1-\alpha&1\\1&-\alpha\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 0$. A nonzero solution is $\begin{bmatrix}\alpha\\1\end{bmatrix}$. Check: $\begin{bmatrix}1-\alpha&1\\1&-\alpha\end{bmatrix}\begin{bmatrix}\alpha\\1\end{bmatrix} = \begin{bmatrix}(1-\alpha)\alpha+1\\\alpha-\alpha\end{bmatrix} = \begin{bmatrix}1+\alpha-\alpha^2\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

Eigenvector for β: $Av = \beta v$ i.e. $(A - \beta I)v = 0$. Take $\begin{bmatrix}\beta\\1\end{bmatrix}$.

$B = \begin{bmatrix}\alpha&\beta\\1&1\end{bmatrix}$ has the eigenvectors as its columns. $AB = \left[A\begin{bmatrix}\alpha\\1\end{bmatrix} \middle| A\begin{bmatrix}\beta\\1\end{bmatrix}\right] = \begin{bmatrix}\alpha^2&\beta^2\\\alpha&\beta\end{bmatrix} = \begin{bmatrix}\alpha&\beta\\1&1\end{bmatrix}\begin{bmatrix}\alpha&0\\0&\beta\end{bmatrix} = BD$, $D = \begin{bmatrix}\alpha&0\\0&\beta\end{bmatrix}$.

Diagonalizing A gives $D = \begin{bmatrix}\alpha&0\\0&\beta\end{bmatrix}$. $\quad A\cancel{BB^{-1}} = BDB^{-1}$ i.e. $\boxed{A = BDB^{-1}}$ $\quad D^n = \begin{bmatrix}\alpha&0\\0&\beta\end{bmatrix}^n = \begin{bmatrix}\alpha^n&0\\0&\beta^n\end{bmatrix}$

$A^n = \underbrace{(BD\cancel{B^{-1}})(\cancel{B}D\cancel{B^{-1}})(\cancel{B}D\cancel{B^{-1}})\cdots(\cancel{B}DB^{-1})}_{n \text{ times}} = BD^nB^{-1}$

$B = \begin{bmatrix}\alpha&\beta\\1&1\end{bmatrix}$ $\quad B^{-1} = \frac{1}{\sqrt5}\begin{bmatrix}1&-\beta\\-1&\alpha\end{bmatrix}$

$\det B = \alpha - \beta = \sqrt5$

$$A^n = BDB^{-1} = \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix}\begin{bmatrix} 1 & -\beta \\ 1 & \alpha \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} \alpha^{n+1}-\beta^{n+1} & \boxed{\alpha\beta^{n+1}-\alpha^{n+1}\beta} \leftarrow \alpha^n-\beta^n \\ \underbrace{\alpha^n-\beta^n} & \boxed{\alpha\beta^n-\alpha^n\beta} \leftarrow \alpha^{n-1}-\beta^{n-1} \end{bmatrix}$$

$$v_n = A^n\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} \alpha^{n+1}-\beta^{n+1} \\ \alpha^n-\beta^n \end{bmatrix}$$

$\alpha\beta = -1$

$\alpha+\beta = 1$

$\alpha-\beta = \sqrt{5}$

$$\alpha\beta = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) = \frac{1-5}{4} = \frac{-4}{4} = -1$$

so $\boxed{F_n = \frac{\alpha^n-\beta^n}{\sqrt{5}}} = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$ grows exponentially

(faster than power law n^k)

eg. $F_0 = \dfrac{\alpha^0-\beta^0}{\sqrt{5}} = \dfrac{1-1}{\sqrt{5}} = 0$

$F_1 = \dfrac{\alpha-\beta}{\sqrt{5}} = \dfrac{\sqrt{5}}{\sqrt{5}} = 1$

$F_2 = \dfrac{\alpha^2-\beta^2}{\sqrt{5}} = \dfrac{(\alpha+1)-(\beta+1)}{\sqrt{5}} = \dfrac{\sqrt{5}}{\sqrt{5}} = 1$

$F_3 = 2$ etc.

$F_{30} = \dfrac{\alpha^{30}-\beta^{30}}{\sqrt{5}} = 832040$