Linear Algebra

Book 2

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$f(x,y) = (3x+2y\ ,\ x-5y)$ can be represented as a matrix transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{bmatrix} 3 & 2 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x+2y \\ x-5y \end{bmatrix}$$

Every linear operator can be expressed as matrix multiplication

eg. consider solutions of $y''+y=0$ i.e. $f(x) = \underbrace{a \sin x + b\cos x}_{\begin{bmatrix} a \\ b \end{bmatrix}}$

$$Df(x) = \underbrace{a\cos x - b\sin x}_{\begin{bmatrix} -b \\ a \end{bmatrix}}$$

$$D(rf+sg) = rDf + sDg$$

$$(rf+sg)' = rf' + sg'$$

$$\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{M}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$$

$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$M^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$M^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$M^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Every 2×2 real matrix A represents a linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ which is the matrix transformation $T_A\begin{bmatrix}x\\y\end{bmatrix} = A\begin{bmatrix}x\\y\end{bmatrix}$.

eg. $\underbrace{\begin{bmatrix}0&-1\\1&0\end{bmatrix}}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-y\\x\end{bmatrix}$ T_A is a counter-clockwise 90° rotation about the origin in $\mathbb{R}^2$:

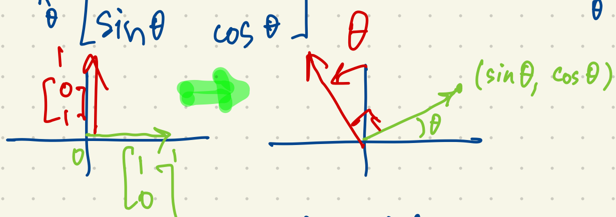
$$T_A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$T_A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$$

$T_A^4 = I$ $\qquad I\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$

A counterclockwise rotation by angle θ about the origin in $\mathbb{R}^2$ represented by the matrix $R_\theta = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

$$R_\theta\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \qquad R_\theta\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

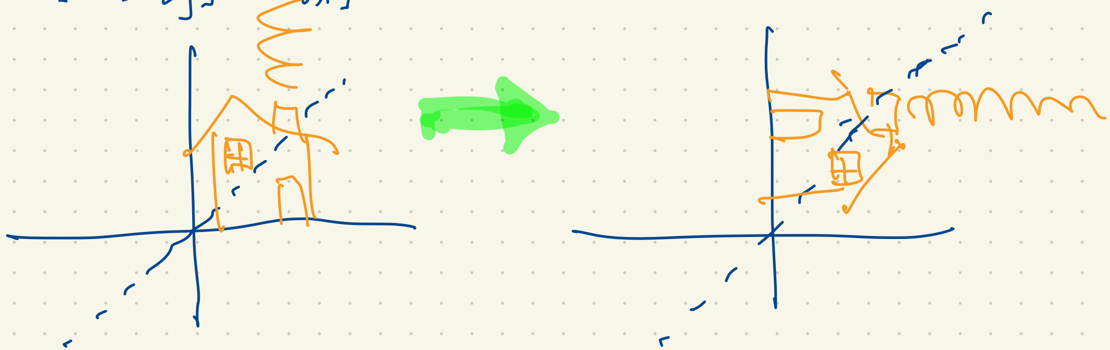


$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$R_\beta R_\alpha = R_{\alpha+\beta}$

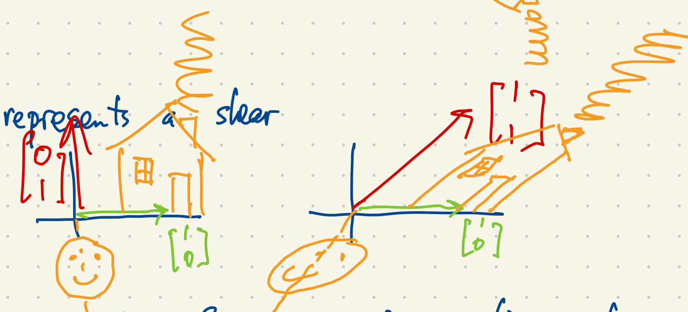
$$\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix} = \begin{bmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{bmatrix}$$

Eg. $\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}y\\x\end{bmatrix}$ is a reflection about the line $y=x$



$\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ represents a reflection in the x-axis

$\begin{bmatrix}1&1\\0&1\end{bmatrix}$ represents a shear



Every matrix transformation is a linear transformation: it takes 0 to 0 and it takes lines to lines or points. It may distort distances and angles.

Example of a "somewhat" generic linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$:

Every linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ takes 0 to 0,

takes lines to lines or points

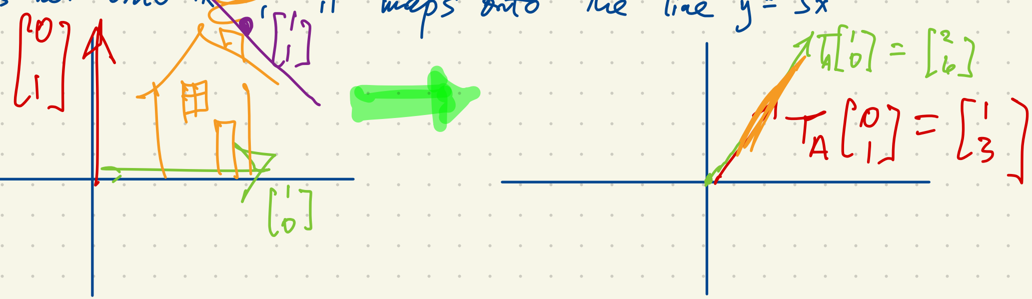
A function $f : A \to B$ is "one-to-one" if $f(x) = f(y)$ implies $x = y$. (No two inputs give the same output.)

f is "onto" if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$.

e.g. $A = \begin{bmatrix}2 & 1\\6 & 3\end{bmatrix}$ defines a linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$, $T_A\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = A\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x+y\\6x+3y\end{bmatrix}$.

This function is not one-to-one e.g. $T_A\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = T_A\left(\begin{bmatrix}-1\\5\end{bmatrix}\right) = \begin{bmatrix}3\\9\end{bmatrix}$

And T_A is not onto $\mathbb{R}^2$; it maps onto the line $y = 3x$

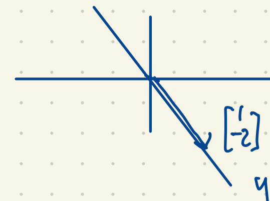


The <u>null space</u> of a linear transformation $\mathrm{Nul}\, T = \{\underline{v} : T\underline{v} = \underline{0}\}$. (the set of null vectors of T).

Recall : $T\underline{0} = \underline{0}$

$$\mathrm{Nul} \underbrace{\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}}_{A} = \mathrm{Nul}\, T_A = \left\{ \begin{bmatrix} x \\ -2x \end{bmatrix} : x \in \mathbb{R} \right\}$$

$$A \begin{bmatrix} x \\ -2x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$



T is one-to-one iff $\mathrm{Nul}\, T = \{\underline{0}\}$ (the only null vector is $\underline{0}$).

This statement should be clear :

On the one hand, suppose T is one-to-one. If $\underline{v} \in \mathrm{Nul}\, T$ then $T\underline{v} = \underline{0} = T\underline{0}$ then $\underline{v} = \underline{0}$.

This says: if T is one-to-one then $\mathrm{Nul}\, T = \{\underline{0}\}$

Conversely, suppose $\mathrm{Nul}\, T = \{\underline{0}\}$. If $T\underline{v} = T\underline{w}$ then $T(\underline{v} - \underline{w}) = T\underline{v} - T\underline{w} = \underline{0}$

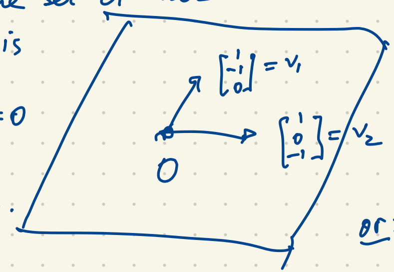
so $\underline{v} - \underline{w} \in \mathrm{Nul}\, T$ i.e. $\underline{v} - \underline{w} = \underline{0}$ i.e. $\underline{v} = \underline{w}$.

"<u>Span</u>" can be used as a noun or as a verb.

The <u>span</u> of a list of vectors $v_1, \dots, v_k$ is the set of all linear combinations of $v_1, \dots, v_k$.

eg. the <u>span</u> of the vectors $\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ in $\mathbb{R}^3$ is the plane $x + y + z = 0$ in $\mathbb{R}^3$

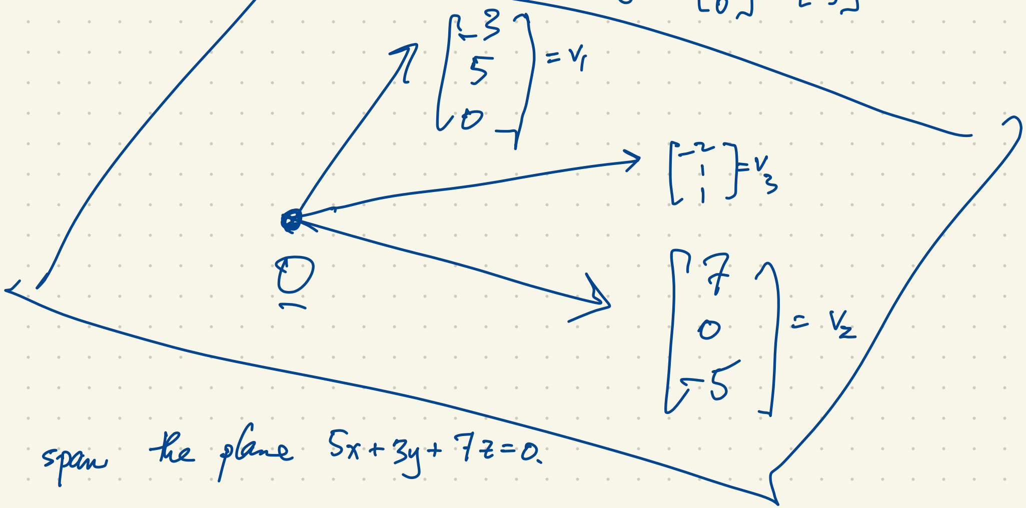
i.e. the plane $z = -x - y$.



We say that the span of v_1 and v_2 is the plane $x + y + z = 0$.

<u>or</u>: v_1 and v_2 <u>span</u> the plane $x + y + z = 0$.

eg. the plane $5x+3y+7z=0$ is spanned by $\begin{bmatrix}-3\\5\\0\end{bmatrix}, \begin{bmatrix}7\\0\\-5\end{bmatrix}$



v_1, v_2, v_3 span the plane $5x+3y+7z=0$.

Given any set of vectors $S \subset \mathbb{R}^3$, the <u>span</u> of S (denoted $\text{span}\, S = \{$linear combinations of vectors in $S\}$) is either $\{\underline{0}\}$ or a line through $\underline{0}$, or a plane through $\underline{0}$, or $\mathbb{R}^3$.

Friday: Quiz 5 on <u>Span</u>.

The <u>image</u> of T is $\{T_A \underline{v} : \underline{v} \in \text{domain of } T_A\}$ is the span of the columns of A.

Eg. $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ defines a linear transformation $T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

(here $\mathbb{R}^3$ consists of 3×1 column vectors)

$$T_A(v) = A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y-z \\ -x+z \\ x-y \end{bmatrix}$$

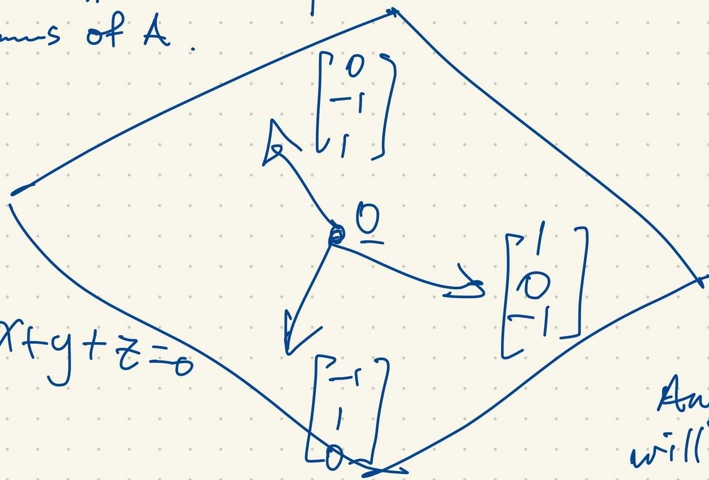
$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

The image of T_A is $\{T_A v : v \in \mathbb{R}^3\} = \left\{ \underbrace{\begin{bmatrix} y-z \\ -x+z \\ x-y \end{bmatrix}} : x,y,z \in \mathbb{R} \right\}$

$$x\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + z\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

(a linear combination of the columns of A)

The image of T_A is the span of the columns of A.



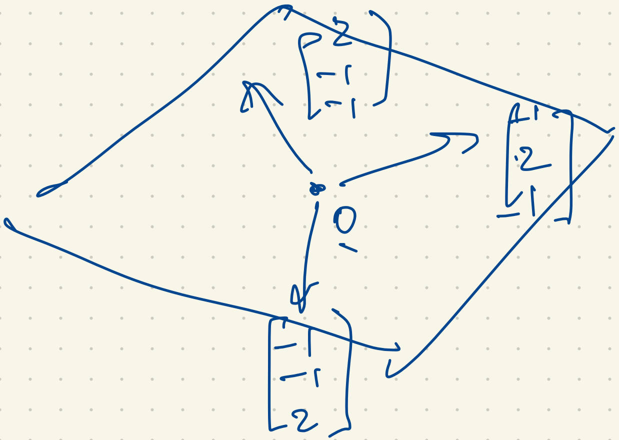
$x+y+z=0$

T_A is <u>not</u> onto $\mathbb{R}^3$. This happens because the columns of A fail to span $\mathbb{R}^3$.

Any 3 <u>linearly</u> independent vectors in $\mathbb{R}^3$ will span all of $\mathbb{R}^3$ (their span is $\mathbb{R}^3$).

Another example: $B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ defines a linear transformation $T_B : \mathbb{R}^3 \to \mathbb{R}^3$.

Once again T_B is not onto $\mathbb{R}^3$; its image is the span of the columns of B i.e. the plane $x+y+z=0$ through the origin in $\mathbb{R}^3$.

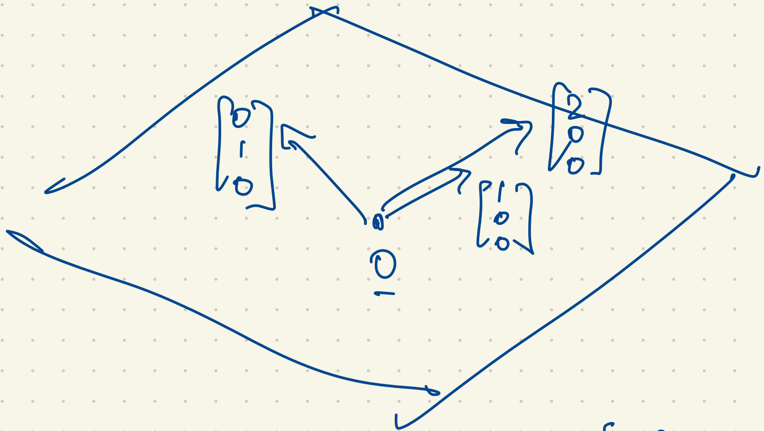


$C = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ has three linearly independent columns spanning $\mathbb{R}^3$ i.e. the image of T_C is $\mathbb{R}^3$ i.e. T_C is onto $\mathbb{R}^3$.

Check: If $a\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + c\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3a-b-c \\ -a+2b-c \\ -a-b+2c \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has $\left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$ as the span of its columns.

T_A is not onto.



The span of the rows of A is $\{ [a,\ 2a,\ b] : a, b \in \mathbb{R} \}$

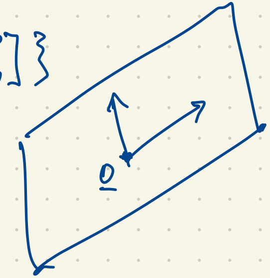
A subspace of $\mathbb{R}^n$ generalizes the notion of $\{\underline{0}\}$, line through the origin, plane through the origin, etc. up to and including $\mathbb{R}^n$ itself. The dimension of such a subspace is $0, 1, 2, 3, \dots, n$.

Given any set $S \subset \mathbb{R}^n$ (any set of vectors) then $\operatorname{span} S = \{$linear combinations of vectors in $S\}$ is a subspace of $\mathbb{R}^n$. Another way is to solve any homogeneous linear system in n variables. The latter case is the same thing as finding the null space of a linear transformation.

In particular if A is an $m \times n$ matrix then $\operatorname{Nul} A = \{ \underline{v} \in \mathbb{R}^n : A\underline{v} = \underset{\text{in } \mathbb{R}^m}{\underline{0}} \}$ is a subspace of $\mathbb{R}^n$.

Eg. a 2-dimensional subspace of $\mathbb{R}^3$ (i.e. a plane through the origin) can be described in either of two ways.

$U = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \right\}$



$x + 3y - z = 0$

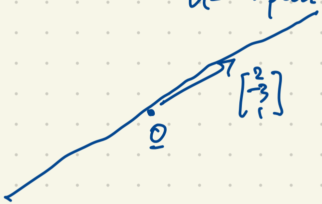
Alternatively, $U = \text{Nul}\begin{bmatrix} 1 & 3 & -1 \end{bmatrix}$

$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \right\}$

$= \left\{ s\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$

Eg. a 1-dimensional subspace of $\mathbb{R}^3$ (i.e. a line through the origin).

$U = \text{span}\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right\}$



$U = \text{Nul}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

i.e. $\begin{cases} x + y + z = 0 \\ x + 2y + 4z = 0 \end{cases}$

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$

$U = \text{Nul}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ $\quad$ x, y are basic variables; z is a free variable.

$z = t$ where t is arbitrary; solve for y, x

$y = -3t$

$x = 2t$

$U = \left\{ \begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\{ t\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

The solutions of $y''+y=0$ form a vector space $\{y : y''+y=0\} = \text{span}\{\sin x, \cos x\}$

$= \{a\sin x + b\cos x : a,b \in \mathbb{R}\}$

$= \text{Nul}\, T$.

Here $Ty = y''+y$ is a function mapping one function to another.

$$T : \{\text{functions}\} \longrightarrow \{\text{functions}\}$$

T is a linear transformation since $T(ay_1+by_2) = aTy_1 + bTy_2$.

Let $T : V \to W$ be a linear transformation.

T is one-to-one iff $\text{Nul}\, T = 0$.

T is onto iff every $w \in W$ has the form $w = Tv$ for some $v \in V$.

T is bijective iff it is both one-to-one and onto. Such functions T have an inverse T^{-1}.

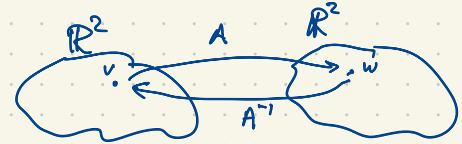
T^{-1} must also be linear.

Eg. consider the 2x2 matrix $A = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$ which represents a linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$.

Find the inverse matrix A^{-1}.

$A^{-1}(Av) = v$ $\qquad$ $A(A^{-1}w) = w$

$A^{-1}A = I$ $\qquad$ $AA^{-1} = I$



$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ identity

Fri. Oct 13 Quiz : Inverses of Matrices

A 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $ad-bc \neq 0$, in which case $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Eg. for $A = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$ we have $3\cdot 5 - 2\cdot 8 = -1$, $A^{-1} = \frac{1}{-1}\begin{bmatrix} 5 & -2 \\ -8 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 8 & -3 \end{bmatrix}$.

Check: $AA^{-1} = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^{-1}A = I$.

Eg. $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$. Compute B^{-1}.

General method: To compute A^{-1}, if it exists, write down $\overbrace{[\underset{n\times n}{A} \mid I_n]}^{n\times 2n}$ and row reduce leading to $\underbrace{[I \mid A^{-1}]}_{n\times 2n}$

$I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

In our case $[B \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 2 & 8 & -1 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 2 & 8 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -\frac{5}{2} & 4 & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right]$$

$B^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$

Check: $B^{-1}B = \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

If the pivots are not all in the leftmost n columns, we don't get I_n on the left. In this case A is not invertible.

Eg. $A = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$.

$$[A \mid I] = \left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 8 & 5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ -1 & -1 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 1 & 1 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 0 & -1 & -8 & 3 \\ 1 & 1 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 3 & -1 \\ 0 & -1 & -8 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 1 & 3 & -1 \\ 0 & 1 & 8 & -3 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 8 & -3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -5 & 2 \\ 8 & -3 \end{bmatrix}$$

Eg. $A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$ has $3 \cdot 2 - 1 \cdot 6 = 0$ so A is not invertible. What goes wrong in our algorithm?

$$[A \mid I] = \left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 6 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right]$$

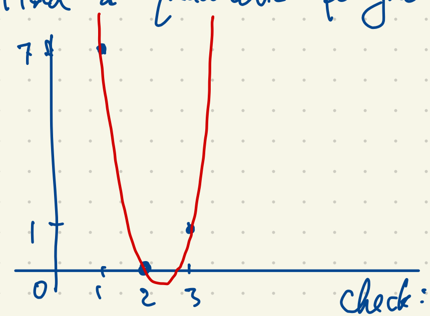
The pivots do not appear in the leftmost two columns so we conclude that A is not invertible.

The image of T_A is the span of the columns of A, namely $\operatorname{span}\left\{ \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$, not $\mathbb{R}^2$. So T_A is not invertible i.e. A is not invertible.

Eg. Find a quadratic polynomial $f(t) = at^2 + bt + c$ having table of values

t	$f(t)$
1	7
2	0
3	1

$$= c + bt + at^2$$



$$f(1) = c + b + a = 7$$
$$f(2) = c + 2b + 4a = 0$$
$$f(3) = c + 3b + 9a = 1$$

Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ -19 \\ 4 \end{bmatrix}$$

So $f(t) = 22 - 19t + 4t^2$

Check: $f(1) = 7$, $f(2) = 0$, $f(3) = 1$ ✓

The solution of a linear system $A\underline{x}=\underline{b}$ is $\underline{x}=A^{-1}\underline{b}$ assuming A is an invertible $n\times n$ matrix.

$[A\,|\,I] \sim \cdots \sim [I\,|\,A^{-1}]$

$A=\begin{bmatrix}3&1\\6&2\end{bmatrix}$ is not invertible since the span of its colums is $\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ ie. A has linearly dependent colums. $\begin{bmatrix}3\\6\end{bmatrix}=3\begin{bmatrix}1\\2\end{bmatrix}$.

Alternatively, A has a null vector $\begin{bmatrix}1\\-3\end{bmatrix}\in \text{Nul}\,A$ since $A\begin{bmatrix}1\\-3\end{bmatrix}=\begin{bmatrix}3&1\\6&2\end{bmatrix}\begin{bmatrix}1\\-3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}=A\begin{bmatrix}0\\0\end{bmatrix}$.

$\text{Nul}\,A=\text{span}\left\{\begin{bmatrix}1\\-3\end{bmatrix}\right\}$ so A is not one-to-one.

The linear system $Ax=\begin{bmatrix}0\\0\end{bmatrix}$ has many solutions.

The linear system $Ax=\begin{bmatrix}1\\7\end{bmatrix}$ has no solutions. since $\begin{bmatrix}1\\7\end{bmatrix}\notin \text{Span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

In 5th edition, I'm omitting
- 2.4 Partitioned Matrices
- 2.5 Matrix Factorizations
- 2.6 Leontief Input/Output Model
- 2.7 Computer graphics

Continue with 2.8 : Subspaces of $\mathbb{R}^n$

A subspace of $\mathbb{R}^n$ is a subset $U\subseteq\mathbb{R}^n$ such that

(i) $\underline{0}\in U$

(ii) For all $u,v\in U$, $u+v\in U$.

(iii) For all $u\in U$ and scalar $c\in\mathbb{R}$, $cu\in U$.

Eg. In $\mathbb{R}^2$, $\{(x,y) : x,y\geq 0\}$ is not a subspace.

Think of: $\{0\}$, line through the origin, plane through the origin, etc.

$U_1\cap U_2=\{u : u\in U_1 \text{ and } u\in U_2\}$

If U_1, U_2 are subspaces of $\mathbb{R}^n$, is $U_1\cap U_2$ also a subspace of $\mathbb{R}^n$?

(i) Since $0\in U_1$ and $0\in U_2$, $0\in U_1\cap U_2$.

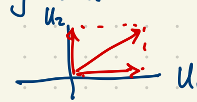
(ii) Let $u,v\in U_1\cap U_2$. Then $u+v\in U_1$ and $u+v\in U_2$ so $u+v\in U_1\cap U_2$.

(iii) Let c be a scalar and $u\in U_1\cap U_2$. Then $cu\in U_1$ and $cu\in U_2$ so $cu\in U_1\cap U_2$.

So yes, the intersection of two subspaces is a subspace.

$U_1 \cup U_2 = \{u : u \in U_1 \text{ or } u \in U_2\}$ i.e. u is in at least one of U_1 or U_2, possibly both.

If U_1 and U_2 are subspaces of $\mathbb{R}^n$, must $U_1 \cup U_2$ also be a subspace? No.



eg. $U_1 = \operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ = the x-axis in $\mathbb{R}^2$

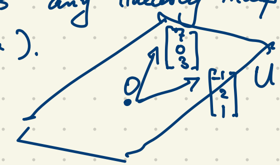
$U_2 = \operatorname{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$ = ... y-axis in $\mathbb{R}^2$

$$\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\text{in } U_1 \cup U_2} + \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\text{in } U_1 \cup U_2} = \begin{bmatrix}1\\1\end{bmatrix} \notin U_1 \cup U_2 .$$

Alternatively, a subspace is a nonempty subset $U \subseteq \mathbb{R}^n$ such that linear combinations of vectors in U is still in U i.e. $\operatorname{span} U = U$.

If U is a subspace of $\mathbb{R}^n$ ($U \leq V$) then a basis for U is any linearly independent set of vectors spanning U.

eg. in $\mathbb{R}^3$, let U be the plane $3x+5y-7z=0$ (through the origin).



The list of vectors $\begin{bmatrix}7\\0\\3\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}$ is a basis for U. These two vectors are linearly independent by inspection. Moreover $\operatorname{span}\left\{\begin{bmatrix}7\\0\\3\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}\right\} = U$ (this is not quite obvious but we will soon see why it's true).

Another basis for U is $\left\{\begin{bmatrix}0\\7\\5\end{bmatrix}, \begin{bmatrix}-5\\3\\0\end{bmatrix}\right\}$

The dimension of U is 2 because we have a basis consisting of 2 vectors.

Syllabus

HW	10
Quizzes	20
Test 1	20
Test 2	20
Exam	30
	100

HW	30
Quizzes	20
Test	20
Exam	30
	100

I'll correct this online and email everyone with this correction.

How do we find a basis for a subspace of $\mathbb{R}^n$?

Eg. If A is an $m\times n$ matrix, $\text{Row } A = \text{span}(\text{rows of } A) \le \mathbb{R}^n$ (really $1\times n$ vectors)

$\text{Col } A = \text{Span}(\text{columns of } A) \le \mathbb{R}^m$ (really $m\times 1$ vectors)

(the row space and column space of A).

Take e.g. $A = \begin{bmatrix} 0 & 1 & -1 & 0 & 3 & 6 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ in reduced row echelon form.

Row A has basis $(0,1,-1,0,3,6)$, $(0,0,0,1,-5,2)$ so Row A is 2-dimensional : $\dim(\text{Row } A) = 2$.

The dimension of $U \le \mathbb{R}^n$ is the number of vectors in a basis for U.

Col A has basis $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$.

$$\text{Col } A = \text{Span}(\text{columns of } A)$$
$$= \left\{ c_1\begin{bmatrix}0\\0\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\0\end{bmatrix} + c_3\begin{bmatrix}-1\\0\\0\end{bmatrix} + c_4\begin{bmatrix}0\\1\\0\end{bmatrix} + c_5\begin{bmatrix}3\\-5\\0\end{bmatrix} + c_6\begin{bmatrix}6\\2\\0\end{bmatrix} : c_1, c_2, \dots, c_6 \text{ any scalars} \right\}$$
$$= \left\{ c_2\begin{bmatrix}1\\0\\0\end{bmatrix} + c_4\begin{bmatrix}0\\1\\0\end{bmatrix} : c_2, c_4 \text{ scalars} \right\} = \left\{ \begin{bmatrix}x\\y\\0\end{bmatrix} : x, y \in \mathbb{R} \right\} \text{ (the } xy\text{-plane)}$$

$\dim \text{Col } A = 2$.

Although row vectors have length 6 and column vectors have length 3, the row space and column space have the same dimension. (equal to the number of pivots).

What if A is not in reduced row echelon form?

Eg. $B = \begin{bmatrix} 0 & 2 & -2 & 1 & 1 & 14 \\ 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 1 & -1 & -2 & 13 & 2 \end{bmatrix}$

Row $B \leq \mathbb{R}^6$
Col $B \leq \mathbb{R}^3$

$$B \sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 2 & -2 & 1 & 1 & 14 \\ 0 & 1 & -1 & -2 & 13 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 0 & 0 & -5 & 25 & -10 \\ 0 & 1 & -1 & -2 & 13 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 1 & -1 & -2 & 13 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & -5 & 25 & -10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 1 & -5 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 3 & -12 & 12 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 0 & 3 & 6 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A$$

Row B = Row A has basis $(0,1,-1,0,3,6)$, $(0,0,0,1,-5,2)$

Col $B \neq$ Col A <u>but</u> Col B has basis $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

In general, the pivot columns of A (= reduced row echelon form of B) tell us which columns of B give a basis ~~for~~ col B.

e.g. $\begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 1 \\ -12 \\ 13 \end{bmatrix} = (\quad) \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + (\quad) \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$