Vector Spaces: Chapter 4

Scalars: real numbers / complex numbers / rational numbers / general fields

A field is a set of scalars in which we can add, subtract, multiply and divide.

A <u>vector space</u> is a set V whose elements are called vectors, including a zero vector $\underline{0}$, and operations $+, -,$ scalar multiplication satisfying

(scalar + scalar = scalar, ~~scalar + vector~~)

~~vector × vector~~

1. For $\underline{u}, \underline{v} \in V$, $\underline{u}+\underline{v} \in V$. (vector + vector = vector)
2. $\underline{u}+\underline{v} = \underline{v}+\underline{u}$
3. $(\underline{u}+\underline{v})+\underline{w} = \underline{u}+(\underline{v}+\underline{w})$
4. $\underline{u}+\underline{0} = \underline{u} = \underline{0}+\underline{u}$

   } For all $\underline{u}, \underline{v}, \underline{w} \in V$

5. For each $\underline{u} \in V$, there is a vector $-\underline{u} \in V$ such that $\underline{u}+(-\underline{u}) = \underline{0}$
6. Scalar multiplication: For every scalar c and $\underline{u} \in V$, $c\underline{u} \in V$ (scalar × vector = vector)
7. Distributivity: $c(\underline{u}+\underline{v}) = c\underline{u}+c\underline{v}$
8. " $(c+d)\underline{v} = c\underline{v}+d\underline{v}$
9. Associativity: $(cd)\underline{v} = c(d\underline{v})$
10. $1\underline{u} = \underline{u}$

$0\underline{u} = \underline{0}$ as follows from the axioms: $0\underline{u}+0\underline{u} = (0+0)\underline{u} = 0\underline{u}$. Add $-0\underline{u}$ to both sides:

scalar, vector

$$(0\underline{u}+0\underline{u})+(-0\underline{u}) = 0\underline{u}+(-0\underline{u}) = \underline{0}$$

By (3), $0\underline{u}+(0\underline{u}+(-0\underline{u})) = \underline{0}$

By (5) $0\underline{u}+\underline{0} = \underline{0}$

$$0\underline{u} = \underline{0}$$

Examples of vector spaces:

$\mathbb{R}^n$ (actually, $\mathbb{R}^{n\times 1}$ is column vectors of length n; $\mathbb{R}^{1\times n}$ is row vectors of length n).

Subspaces of $\mathbb{R}^n$

The set of all polynomials of degree $< n$ in x is an n-dimensional vector space

$$V = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} : a_0, a_1, a_2, \ldots, a_{n-1} \text{ are scalars}\}.$$

$\{1, x, x^2, \ldots, x^{n-1}\}$ is a basis for V. x is an indeterminate (i.e. not a number, just a symbol).

$\{1, x, x(x-1), x(x-1)(x-2), \ldots, x(x-1)(x-2)\cdots(x-n+1)\}$ is also a basis.

The set of all polynomials in x is a vector space which is infinite-dimensional.

A basis is $\{1, x, x^2, x^3, x^4, \ldots\}$

Examples of polynomials: $5-3x+2x^2$, $1-x^3+3x^7+11x^8$, ...

Not polynomials: $\sin x$, $\sqrt{1+x}$, $x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$

The set of all functions $\mathbb{R} \to \mathbb{R}$.

As a subspace of this, the continuous functions $\mathbb{R} \to \mathbb{R}$.

An even smaller subspace: differentiable functions $\mathbb{R} \to \mathbb{R}$.

Even smaller: the space of "smooth functions" $V = \{f : \mathbb{R} \to \mathbb{R} : f^{(n)} \text{ exists for all } n \geq 0\}$ (nth derivative)

A linear transformation $T : V \to V$ is defined by $T = D^2 + I$ $\quad (D = \frac{d}{dx})$ i.e. $Tf = f'' + f$.

The rank of T is infinite dimensional. T is not one-to-one.

A basis for $\operatorname{Nul} T = \{f : Tf = 0\}$ is $\{\sin x, \cos x\}$. $\quad Tf = 0$ iff $f(x) = a\sin x + b\cos x$ for some $a, b \in \mathbb{R}$.

$D : V \to V$ has $\operatorname{Nul} D = \{\text{constant functions}\}$ having basis $\{1\}$; $\operatorname{Nul} D$ is one-dimensional.

D has eigenvectors! eg. $De^{3x} = 3e^{3x}$. For every $\lambda \in \mathbb{R}$, the set of eigenvectors having eigenvalue λ is one-dimensional with basis $\{e^{\lambda x}\}$.