Linear Algebra

Book 3

Eg. $A = \begin{bmatrix} 1 & 3 & 0 & 41 \\ 2 & 4 & 11 & 7 \\ 0 & 3 & 0 & 4 \\ 1 & 6 & 3 & 5 \end{bmatrix}$

Expanding along the third row, $\det A = \cancel{0\begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}} - 3\begin{vmatrix} 1 & 0 & 41 \\ 2 & 11 & 7 \\ 1 & 3 & 5 \end{vmatrix} + \cancel{0\begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}} - 4\begin{vmatrix} 1 & 3 & 0 \\ 2 & 4 & 11 \\ 1 & 6 & 3 \end{vmatrix}$

$$= -3\left(\begin{vmatrix} 11 & 7 \\ 3 & 5 \end{vmatrix} + 41\begin{vmatrix} 2 & 11 \\ 1 & 3 \end{vmatrix}\right) - 4\left(\begin{vmatrix} 4 & 11 \\ 6 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 11 \\ 1 & 3 \end{vmatrix}\right)$$

$$= -3\left(55-21+41(6-11)\right) - 4\left(12-66-3(6-11)\right)$$

$$= 669.$$

(I checked this by computer.)

Wed. Nov. 8 Test. Come a few minutes early if you can.
No Quiz Fri. Nov 10, 17.
I am away Fri. Nov. 17, Mon Nov 20. Lectures for those two days will be prerecorded - check the websites.

Recall: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = ad-bc$. A is invertible iff $\det A \neq 0$, in which case $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

This formula has a generalization for $n\times n$ matrices (Cramer's Rule). This is useful although not the most computationally efficient way to compute A^{-1} if n is large.

On HW 2 you had to find A^{-1} where A is 4×4. The entries of A^{-1} have a common denominator $6 = \det A$.

Eg. $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 3 \\ 7 & 6 & 4 \end{bmatrix}$, $\det A = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 1 & 3 \\ 7 & 6 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & -3 & -7 \\ 7 & 6 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & 7 \\ 7 & 6 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & 7 \\ 0 & -8 & -31 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & 7 \\ 0 & 8 & 31 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & 7 \\ 0 & -1 & 10 \end{vmatrix}$

$$= |1| \begin{vmatrix} 3 & 7 \\ -1 & 10 \end{vmatrix} = 1 \cdot 37.$$

A^{-1} has fractional entries with common denominator 37.

Matrix of minors: $M = \begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 6 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 7 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 7 & 6 \end{vmatrix} \\ \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 7 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 7 & 6 \end{vmatrix} \\ \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -14 & -13 & 5 \\ -22 & -31 & -8 \\ 1 & -7 & -3 \end{bmatrix}$

$A^{-1} = \frac{1}{37} \begin{bmatrix} -14 & 22 & 1 \\ 13 & -31 & 7 \\ 5 & 8 & -3 \end{bmatrix}$

transpose;
apply checkerboard;
divide by det A

$$= \begin{bmatrix} -\frac{14}{37} & \frac{22}{37} & \frac{1}{37} \\ \frac{13}{37} & -\frac{31}{37} & \frac{7}{37} \\ \frac{5}{37} & \frac{8}{37} & -\frac{3}{37} \end{bmatrix}$$

Check: $A A^{-1} = \frac{1}{37} \begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 3 \\ 7 & 6 & 4 \end{bmatrix} \begin{bmatrix} -14 & 22 & 1 \\ 13 & -31 & 7 \\ 5 & 8 & -3 \end{bmatrix} = \frac{1}{37} \begin{bmatrix} 37 & 0 & 0 \\ 0 & 37 & 0 \\ 0 & 0 & 37 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

If A is a square matrix with integer entries and $\det A = \pm 1$, then A^{-1} also has integer entries.

Find a constant c such that the following matrix has determinant zero:

$$A = \begin{bmatrix} 5 & 3 & 6 \\ 1 & 2 & 4 \\ 7 & 7 & c \end{bmatrix} \begin{matrix} \leftarrow u = (5\ 3\ 6) \\ \leftarrow v = (1\ 2\ 4) \\ \leftarrow w \quad u + 2v = (7\ 7\ 14) \end{matrix}$$

If $c = 14$ then A has linearly dependent rows so $\det A = 0$ in this case (A is not invertible).

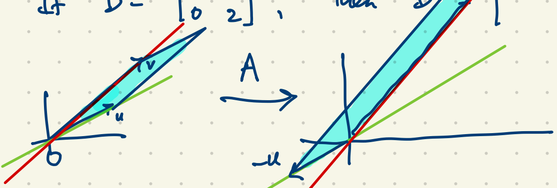
If $c \neq 14$ then A has linearly independent rows then $w \neq (7\ 7\ 14)$ and $(0\ 0\ 1)$ is a linear combination of u, v, w i.e. Row A contains $u, v, (0\ 0\ 1)$.

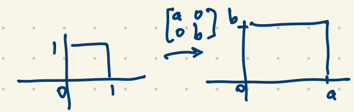
$$\det \left[\begin{array}{cc|c} 5 & 3 & 6 \\ 1 & 2 & 4 \\ \hline 0 & 0 & 1 \end{array}\right] = 7 \times 1 = 7 \neq 0 .$$

If $A = \begin{bmatrix} -25 & 36 \\ -18 & 26 \end{bmatrix}$, then $A^{10} =$

If $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, then $D^{10} = \begin{bmatrix} & \\ & \end{bmatrix}$

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} ac & 0 \\ 0 & bd \end{bmatrix}$$





$\det A = -2$

$$\begin{vmatrix} -25 & 36 \\ -18 & 26 \end{vmatrix} = -25 \times 26 + 36 \times 18 = -2$$

Basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ standard basis

$$\begin{bmatrix} x \\ y \end{bmatrix} = x e_1 + y e_2$$

There is a basis $\{u, v\}$ for $\mathbb{R}^2$ such that $Au = -u$, $Av = 2v$

$$A^{10} u = AAA \cdots Au$$
$$A^2 u = AAu = A(-u) = -Au = u$$
$$A^3 u = AAA u = -u$$
$$\vdots$$
$$A^{10} u = u$$

$$A^2 v = AAv = A(2v) = 2Av = 4v$$
$$A^3 v = 8v$$
$$A^{10} v = \underbrace{1024}_{2^{10}} v$$

u, v are eigenvectors of A with corresponding eigenvalues $-1, 2$.

__Definition__ If A is an $n\times n$ matrix, and $v\in\mathbb{R}^n$, then v is an eigenvector for A with eigenvalue λ if
$Av=\lambda v$.

How do we find eigenvalues and eigenvectors?

If $Av=\lambda v$ then $Av-\lambda v=0$ i.e. $Av-\lambda I v=0$ i.e. $(A-\lambda I)v=0$.
We should assume $v\neq 0$ is a nonzero null vector for $A-\lambda I$. This can only happen if $\det(A-\lambda I)=0$.
This condition allows us to solve for the corresponding eigenvalue λ. Solve for λ; and for each value λ (each eigenvalue), solve $(A-\lambda I)v=0$ for the corresponding eigenvector(s) v.

For $A=\begin{bmatrix}-25 & 36\\ -18 & 26\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}-25 & 36\\ -18 & 26\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}=\begin{bmatrix}-25-\lambda & 36\\ -18 & 26-\lambda\end{bmatrix}$.

$$\begin{vmatrix}-25-\lambda & 36\\ -18 & 26-\lambda\end{vmatrix}=(-25-\lambda)(26-\lambda)+36\times 18=\lambda^2-\lambda-2=(\lambda+1)(\lambda-2)$$

The characteristic polynomial has two roots $\lambda_1=-1$, $\lambda_2=2$ (the two eigenvalues).
To find the corresponding eigenvectors v_1, v_2:
First take $\lambda_1=-1$ and solve $Av_1=\underbrace{-v_1}_{\lambda_1=-1}$ i.e. $(A+I)v_1=0$. $A+I=\begin{bmatrix}-24 & 36\\ -18 & 27\end{bmatrix}\begin{bmatrix}3\\ 2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ (by inspection)
Or $\begin{bmatrix}-24 & 36\\ -18 & 27\end{bmatrix}\sim\begin{bmatrix}1 & -3/2\\ -18 & 27\end{bmatrix}\sim\begin{bmatrix}1 & -3/2\\ 0 & 0\end{bmatrix}$ has null space $\mathrm{Span}\left\{\begin{bmatrix}3/2\\ 1\end{bmatrix}\right\}$ with basis $\begin{bmatrix}3/2\\ 1\end{bmatrix}$

$$\begin{bmatrix}1 & -3/2\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} : \quad \begin{aligned}x-\tfrac{3}{2}y&=0\\ 0&=0\end{aligned}$$

Introduce a parameter t.
$$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}\frac{3}{2}t\\ t\end{bmatrix}=t\begin{bmatrix}3/2\\ 1\end{bmatrix}$$

We can take v_1 to be any nonzero scalar multiple of $\begin{bmatrix}3/2\\ 1\end{bmatrix}$. I'll take $v_1=\begin{bmatrix}3\\ 2\end{bmatrix}$. So $Av_1=\lambda_1 v_1=-v_1$.

For $\lambda_2 = 2$: Solve $Av_2 = \lambda_2 v_2 = 2v_2$ i.e. $(A-2I)v_2 = 0$ where $A - 2I = \begin{bmatrix} -25 & 36 \\ -18 & 26 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -27 & 36 \\ -18 & 24 \end{bmatrix}$

A null vector of $A-2I$: $v_2 = \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ So $\begin{bmatrix} -27 & 36 \\ -18 & 24 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ i.e. $Av_2 = \lambda_2 v_2 = 2v_2$.

$v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is a basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

We started with $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as the standard basis.

Check: A is similar D $(A = BDB^{-1})$

so $\operatorname{tr} A = \operatorname{tr} D$, $\underbrace{\det A}_{-2} = \underbrace{\det D}_{-2}$.

trace of $A = \operatorname{tr} A = 1$, $\operatorname{tr} D = 1$

To find A^{10}: two approaches.

Let $B = [v_1 | v_2] = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Then $AB = A\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 8 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = BD$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ (diagonal matrix)

so $A\cancel{BB^{-1}} = BDB^{-1}$ i.e. $A = BDB^{-1}$.

So $A^{10} = (BDB^{-1})(BDB^{-1}) \cdots (BDB^{-1}) = BD^{10}B^{-1} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -8183 & 12276 \\ -6138 & 9208 \end{bmatrix}$

To check: $\det(A^{10}) = (\det A)^{10} = (-2)^{10} = 1024$. $\det[\ \downarrow\ ] = 1024$.

$\det A = (-25)(26) - (36)(-18) = -2$.

$\det A = (\det B)(\det D)(\det B^{-1}) = 1 \times (-2) \times 1 = -2$

Second approach: $A^{10} v_1 = v_1$, $A^{10} v_2 = 1024 v_2$

$$\left.\begin{aligned} v_1 &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3e_1 + 2e_2 \\ v_2 &= \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 4e_1 + 3e_2 \end{aligned}\right\} \Rightarrow \begin{aligned} e_1 &= 3v_1 - 2v_2 = 3\begin{bmatrix} 3 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ e_2 &= -4v_1 + 3v_2 = -4\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

$$A^{10}e_1 = A^{10}(3v_1 - 2v_2) = 3 \cdot v_1 - 2 \times 1024 v_2 = 3\begin{bmatrix} 3 \\ 2 \end{bmatrix} - 2048\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -8183 \\ -6138 \end{bmatrix}$$

$$A^{10}e_2 = A^{10}(-4v_1 + 3v_2) = -4v_1 + 3 \times 1024 v_2 = -4\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 3072\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 12276 \\ 9208 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} -8183 & 12276 \\ -6138 & 9208 \end{bmatrix}$$

A and D are similar: they represent the same linear transformation with respect to different choices of basis.

Eg. diagonalize the matrix $A = \begin{bmatrix} 4 & -1 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

$\det A = \begin{vmatrix} 4 & -1 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 2 & 1 \end{vmatrix} \cdot 3 = 6 \cdot 3 = 18$

First compute the characteristic polynomial $\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -1 & 1 \\ 2 & 1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} (3-\lambda) = [(4-\lambda)(1-\lambda)+2](3-\lambda)$

$= [\lambda^2 - 5\lambda + 6](3-\lambda) = (\lambda-2)(\lambda-3)(3-\lambda) = -(\lambda-2)(\lambda-3)^2$ has roots $2, 3, 3$ (the eigenvalues of A).

Find eigenvector v_1 for $\lambda_1 = 2$: solve $(A - \lambda_1 I)v_1 = 0$ i.e. $\begin{bmatrix} 2 & -1 & 1 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{v_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $v_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \Rightarrow Av_1 = 2v_1$.

Find eigenvectors v_2, v_3 for $\lambda_2 = \lambda_3 = 3$: solve $(A - 3I)v = 0$ i.e. $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Take $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Note: We want two linearly independent solutions.

Form the matrix $B = [v_1 | v_2 | v_3] = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ whose columns are the eigenvectors. (v_1, v_2, v_3 is our basis of eigenvectors)

Then $AB = BD$ where $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

i.e. $A\cancel{BB^{-1}} = BDB^{-1}$

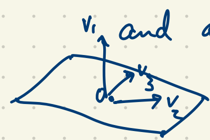
i.e. $A = BDB^{-1}$. We have diagonalized A.

$AB = A[v_1 | v_2 | v_3] = [Av_1 | Av_2 | Av_3] = [2v_1 | 3v_2 | 3v_3] = [v_1 | v_2 | v_3] \begin{bmatrix} 2 & & \\ & 3 & \\ & & 3 \end{bmatrix} = BD$

Check: $\operatorname{tr} A \overset{?}{=} \operatorname{tr} D$, $\det A \overset{?}{=} \det D$

$8 = 8$, $18 = 18$

$\mathbb{R}^3$ has an eigenvector v_1 with eigenvalue $\lambda_1 = 2$ and an eigenspace $\operatorname{Span}\{v_2, v_3\}$ with eigenvalue 3.



$x - y + z = 0$ ($\operatorname{Span}\{v_2, v_3\}$)

The eigenspace for λ is $\mathrm{Nul}\,(A-\lambda I) = \{$all eigenvectors having eigenvalue $\lambda\}$

$= \{$all v satisfying $Av = \lambda v\}$.

$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ has a single eigenspace $\mathbb{R}^3$ with eigenvalue 5.

Actually, we don't necessarily have a basis of eigenvectors.

Consider $A = \begin{bmatrix} -7 & 16 \\ -4 & 9 \end{bmatrix}$.

Find the characteristic polynomial $\det(A-\lambda I) = \begin{vmatrix} -7-\lambda & 16 \\ -4 & 9-\lambda \end{vmatrix} = (-7-\lambda)(9-\lambda) + 64 = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2$

which has roots 1, 1.

Look for eigenvectors: $(A-I)v = 0$ i.e. $\begin{bmatrix} -8 & 16 \\ -4 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Take $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.