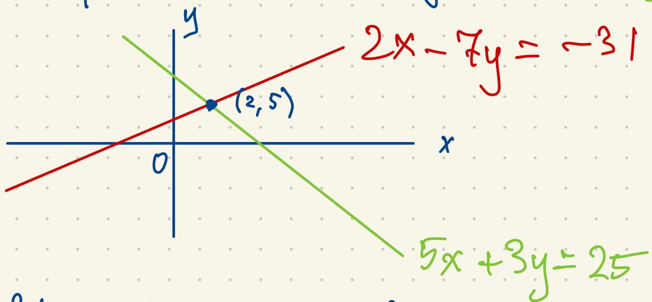
Linear Algebra

Book 1

Example: Find all (x,y) such that $5x+3y=25$ and $2x-7y=-31$.



We are asking for the simultaneous solution of a system of two equations in two unknowns x and y.

$$\begin{cases} 5x+3y=25 & (1) \\ 2x-7y=-31 & (2) \end{cases}$$

$2\times(1) - 5\times(2) = (3)$

$2\times 3 - 5(-7) = 6+35 = 41$

$2\times 25 - 5\times(-31) = 50+155 = 205$

$41y = 205$

$(4) = (3) \div 41$

$y = 5$

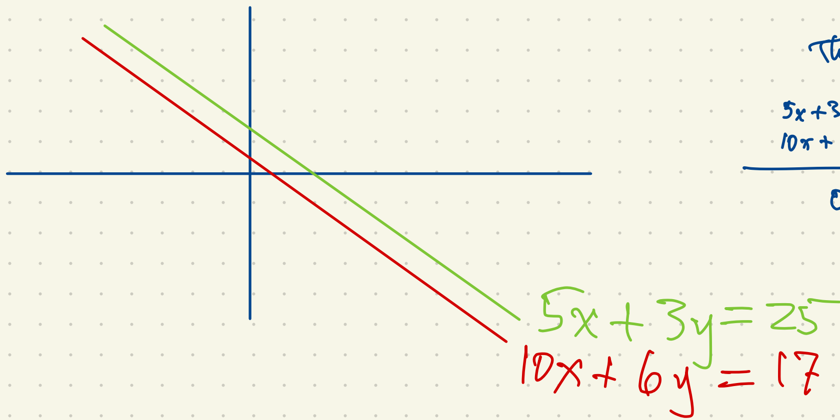
$5x+15 = 25$

$5x = 10$

$x = 2$

Solution: $(x,y) = (2,5)$ is the unique solution.

Example: Find all (x,y) such that $5x+3y=25$ and $10x+6y=17$.



This system is inconsistent: it has no solution.

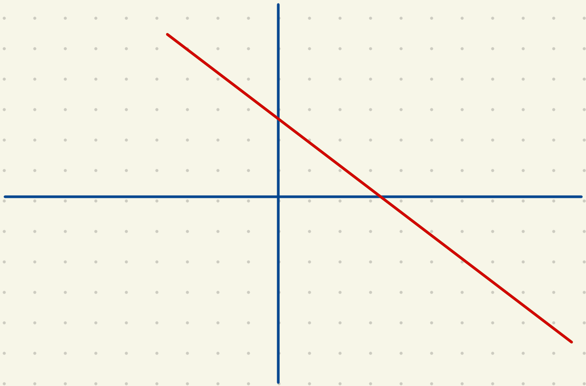
$5x+3y=25 \quad (1)$

$10x+6y=17 \quad (2)$

$0 = 33 \quad 2\times(1)-(2)$

This is inconsistent.

Example : Find all (x,y) such that $5x+3y=25$ and $15x+9y=75$.



$5x+3y=25$

$15x+9y=75$

This system is consistent but the solution is not unique: there are infinitely many solutions.

$$\begin{aligned} 5x+3y&=25 && (1)\\ 15x+9y&=75 && (2)\\ \hline 0&=0 && (3) = 3\times(1)-(2) \end{aligned}$$

A system of m linear equations in n unknowns has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

(a_{ij}, b_i constants for $i \in \{1,\dots,m\}$, $j = \{1,2,\dots,n\}$; $x_1,\dots,x_n$ variables representing unknowns).

Typically, when $m=n$ we can expect a unique solution;

$m>n$ " — — — " : no solution (inconsistent system);

$m<n$ " — — " : more than one solution.

Example with $m=n=3$: a system of 3 linear equations in 3 unknowns.
Kim buys a bag of 26 items weighing 226 oz. costing \$34. The items included

cans of tuna (\$1 each, 5 oz each)
apples (\$1 each, 8 oz each)
loaves of bread (\$3 each, 20 oz each)

How many of each item did Kim buy? (say x cans of tuna, y apples, z loaves of bread)

$$\begin{aligned} x + y + z &= 26 && (1) \\ 5x + 8y + 20z &= 226 && (2) \\ x + y + 3z &= 34 && (3) \end{aligned}$$

$$\begin{aligned} 2z &= 8 && (3)-(1) = (4) \\ z &= 4 && (5) \\ x + y &= 22 && (6) = (8)-(5) \\ 5x + 8y &= 146 && (7) \\ 3y &= 36 && (7) - 5\times(6) = (8) \\ y &= 12 && (9) = (8)\div 3 \\ x &= 10 && (10) = (6)-(9). \end{aligned}$$

$146 - 5\times 22 = 146 - 110 = 36$

The unique solution of this system is $(x,y,z) = (10,12,4)$. (Kim bought 10 cans of tuna, 12 apples, and 4 loaves of bread.)

Check: that all three equations are satisfied.

Matrix formulation of linear systems

$$\begin{aligned} x + y + z &= 26 \\ 5x + 8y + 20z &= 226 \\ x + y + 3z &= 34 \end{aligned} \longrightarrow \begin{array}{cccc} x & y & z & \text{total} \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 1 & 1 & 3 & 34 \end{array}\right]$$

$226 - 5 \times 26 = 226 - 130$
$= 96$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 1 & 1 & 3 & 34 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 0 & 0 & 2 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 0 & 3 & 15 & 96 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 0 & 1 & 5 & 32 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

subtract row 1 from row 3; divide row 3 by 2; subtract 5 times row 1 from row 2; divide row 2 by 3

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 14 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{array}\right] \quad \text{i.e.} \quad \begin{aligned} x &= 10 \\ y &= 12 \\ z &= 4 \end{aligned}$$

subtract 5 times row 3 from row 2; subtract row 2 from row 1; subtract row 3 from row 1

Example: Find all (x, y) such that $5x + 3y = 25$ and $2x - 7y = -31$.

$-7 - \frac{6}{5} = -\frac{35}{5} - \frac{6}{5} = -\frac{41}{5}$

$-31 - 10 = -41$

$$\begin{array}{cc} x & y \end{array} \left[\begin{array}{cc|c} 5 & 3 & 25 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3/5 & 5 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3/5 & 5 \\ 0 & -\frac{41}{5} & -41 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3/5 & 5 \\ 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$$

divide row 1 by 5; subtract 2 times row 1 from row 2; multiply row 2 by $-\frac{5}{41}$; subtract $\frac{3}{5}$ times row 2 from row 1

Solution: $(x, y) = (2, 5)$.

Alternatively:

$$\left[\begin{array}{cc|c} 5 & 3 & 25 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -7 & -31 \\ 5 & 3 & 25 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{7}{2} & -\frac{31}{2} \\ 5 & 3 & 25 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$$

interchange rows 1 and 2

Even better: $\left[\begin{array}{cc|c}5&3&25\\2&-7&-31\end{array}\right] \sim \left[\begin{array}{cc|c}1&17&87\\2&-7&-31\end{array}\right] \sim \left[\begin{array}{cc|c}1&17&87\\0&-41&-205\end{array}\right] \sim \left[\begin{array}{cc|c}1&17&87\\0&1&5\end{array}\right]$

subtract 2 times row 2 from row 1 — subtract 2 times row 1 from row 2 — divide row 2 by -41

$-31-2\times 87$
$=-31-174$
$=-205$

$\sim \left[\begin{array}{cc|c}1&0&2\\0&1&5\end{array}\right]$

subtract 17 times row 2 from row 1

Solution: $(x,y)=(2,5)$.

Check! $5\times 2+3\times 5=25$
$2\times 2-7\times 5=-31$

Elementary row operations:
(i) add a multiple of one row to another
(ii) multiply a row by a nonzero constant
(iii) interchange two rows

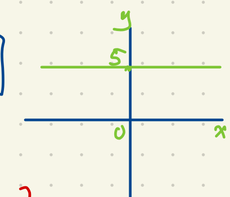
$A\sim B$ means that A,B are linear systems having the same solutions.
We use $\underline{\text{Gaussian elimination}}$ to reduce $A_1\sim A_2\sim\cdots\sim A_m$ where A_1 represents the linear system and A_m represents an equivalent linear system (ie. having the same solutions) but A_m is simpler than A_1. Each step $A_i\sim A_{i+1}$ is obtained by one elementary row operation.
why just one operation at a time?

$$\left.\begin{aligned}5x+3y&=25\\2x-7y&=-31\end{aligned}\right\} \quad \left[\begin{array}{cc|c}5&3&25\\2&-7&-31\end{array}\right] \sim \left[\begin{array}{cc|c}1&\frac{3}{5}&5\\1&-\frac{7}{2}&-\frac{31}{2}\end{array}\right] \not\sim \left[\begin{array}{cc|c}0&\frac{41}{10}&\frac{41}{2}\\0&-\frac{41}{10}&-\frac{41}{2}\end{array}\right] \sim \left[\begin{array}{cc|c}0&1&5\\0&1&5\end{array}\right] \sim \left[\begin{array}{cc|c}0&1&5\\0&0&0\end{array}\right]$$

divide row 1 by 5
divide row 2 by 2

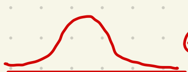
subtract row 2 from row 1
subtract row 1 from row 2

ie. $y=5$
$0=0$



unique solution (2,5)

infinitely many solutions

Gauss  Gaussian distribution

$\left[\begin{array}{cc|c} 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$, $\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 0 & 4 \end{array}\right]$, $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$ are examples of matrices in <u>reduced</u> <u>row</u> <u>echelon form</u> :

they cannot be simplified any further by elementary row operations.

$\left[\begin{array}{cc|c} 1 & 17 & 87 \\ 0 & 1 & 5 \end{array}\right]$ is almost reduced; it is in <u>row echelon form</u> .

For a linear system whose matrix is in row echelon form, we can solve for the unknowns $x_1, x_2, \cdots, x_n$, we solve for x_n, then x_{n-1}, then x_{n-2}, $\cdots$, x_1 by <u>back-substitution</u>.

eg. $\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 5 & 3 & 7 & 3 \\ 0 & 2 & 11 & 4 \\ 0 & 0 & 6 & 8 \end{array}\right] \end{array}$ is in row echelon form.

Every linear system has a unique reduced row echelon form.

In any $m \times n$ matrix, a <u>pivot</u> is the first nonzero entry in its row.
(Pivots are highlighted above.)

In order for a matrix to be in <u>row echelon form</u>, we must have

- pivots in any row must occur to the right of pivots in any previous rows;
- any zero rows occur at the bottom.

Assuming a matrix is already in row echelon form, then to be in <u>reduced</u> <u>row</u> <u>echelon</u> <u>form</u>, we must have

- every pivot entry must be a 1
- every column having a pivot has only one nonzero entry.

Example: Solve the following linear system of 3 equations in 5 unknowns:

$$\begin{cases} x_1 + 4x_2 - x_3 + 2x_4 + 3x_5 = 6 \\ 2x_1 + 8x_2 - x_3 + 7x_4 + 4x_5 = 19 \\ -x_1 - 4x_4 + 4x_3 + 8x_4 - 4x_5 = 26 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 2 & 8 & -1 & 7 & 4 & 19 \\ -1 & -4 & 4 & 8 & -4 & 26 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ -1 & -4 & 4 & 8 & -4 & 26 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 3 & 10 & -1 & 32 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right]$$

This matrix is in row echelon form. This can be used to solve the linear system by back-substitution.

$x_4 + 5x_5 = 11$ $\qquad$ $x_5 = t$ is a free parameter.

$x_4 = 11 - 5t$

$x_3 + 3x_4 - 2x_5 = 7$

$x_3 = 7 - 3x_4 + 2x_5 = 7 - 3(11-5t) + 2t = -26 + 17t$

$x_1 + 4x_2 - x_3 + 2x_4 + 3x_5 = 6$

$x_2 = s$ is another free parameter

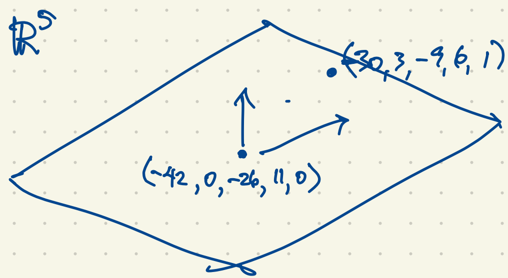
$x_1 = 6 - 4x_2 + x_3 - 2x_4 - 3x_5 = 6 - 4s + (-26+17t) - 2(11-5t) - 3t$

$= -42 - 4s + 24t$

Solutions: $(x_1, x_2, x_3, x_4, x_5) = (-42 - 4s + 24t,\ s,\ -26+17t,\ 11-5t,\ t)$ where s,t are arbitrary.

Geometrically, the set of solutions forms a plane (2-dimensional surface) in $\mathbb{R}^5$.

two parameters s,t are coordinates for the plane



Solution set inside $\mathbb{R}^5$.

The (solution) point corresponding to $(s,t) = (3,1)$ is $(30, 3, -9, 6, 1)$ is another solution.

Our system is consistent but the solution is not unique.

$$\left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 5 & 1 & 13 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 0 & -24 & -42 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 0 & -24 & -42 \\ 0 & 0 & 1 & 0 & -17 & -26 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right]$$

(row echelon form) (reduced row echelon form)

To solve a linear system in reduced row echelon form, introduce parameters for the free variables (the variables whose columns do not contain a pivot).

In the example above, x_2 and x_5 are the free variables. Introduce s, t. $x_2 = s$, $x_5 = t$ can be chosen freely. Solve for the variables x_1, x_3, x_4 using the equations appearing in the reduced row echelon form:

$$\left.\begin{aligned} x_1 + 4s - 24t &= -42 \\ x_3 \quad -17t &= -26 \\ x_4 + 5t &= 11 \end{aligned}\right\} \Rightarrow (x_1, x_2, x_3, x_4, x_5) = (-42 - 4s + 24t,\ s,\ -26 + 17t,\ 11 - 5t,\ t)$$

where s, t are arbitrary.

(This is the parametric solution in terms of the parameters s, t. The system is consistent, having infinitely many solutions.)

As long as the rightmost column has no pivot, the system is consistent.

The general solution can be written as

$$\begin{aligned} (x_1, x_2, x_3, x_4, x_5) &= (-42 - 4s + 24t,\ s,\ -26 + 17t,\ 11 - 5t,\ t) \\ &= (-42, 0, -26, 11, 0) + s(-4, 1, 0, 0, 0) + t(24, 0, 17, -5, 1) \end{aligned}$$

$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$ (vector addition)

$c\,\underbrace{(a_1, a_2, \ldots, a_n)}_{\text{vector}} = (ca_1, ca_2, \ldots, ca_n)$ (scalar multiplication)

c: scalar

Algebraic operations for matrices

If $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix}$ then $BA = \underbrace{\begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \end{bmatrix}}_{2\times 3} = \underbrace{\begin{bmatrix} 13 & 11 & 41 \\ -1 & -27 & 23 \end{bmatrix}}_{2\times 3}$

Here AB is undefined.

An $m\times n$ matrix has the form

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

$a_{i,j}$ is the (i,j)-entry of the matrix A.

$i \in \{1, 2, \dots, m\}$

$j \in \{1, 2, \dots, n\}$.

Often $a_{i,j}$ is written a_{ij}. (unless this results in confusing)

If A is $m\times n$ and B is $n\times r$ then $\underbrace{A}_{m\times n}\underbrace{B}_{n\times r}$ is $m\times r$.

We can't multiply two matrices unless the number of columns in the first matrix equals the number of rows in the second matrix.

eg. $\underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}}_{3\times 2} = \begin{bmatrix} 12 & 8 \\ 23 & 4 \end{bmatrix}$ whereas $\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \end{bmatrix}}_{2\times 3} = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -7 & 11 \\ 5 & -1 & 21 \end{bmatrix}$

If $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$ then $A^2 = \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}}_{A} = \begin{bmatrix} 7 & 3 \\ 1 & 4 \end{bmatrix}$, $A^3 = A^2 A = \begin{bmatrix} 7 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 17 & 18 \\ 6 & -1 \end{bmatrix}$

Recall: the linear system $\left.\begin{array}{l}5x+3y=25\\2x-7y=-31\end{array}\right\}$ has a unique solution $(x,y)=(2,5)$.

One way to solve this: Write the linear system as $Av=b$ where A is a 2×2 matrix, $v=\begin{bmatrix}x\\y\end{bmatrix}$ is a 2×1 matrix (i.e. column vector of length 2) and $b=\begin{bmatrix}25\\-31\end{bmatrix}$ is a 2×1 matrix of constants.

Here $A=\begin{bmatrix}5&3\\2&-7\end{bmatrix}$.

$Av=b$ says $\underbrace{\begin{bmatrix}5&3\\2&-7\end{bmatrix}}_{2\times\cancel{2}}\underbrace{\begin{bmatrix}x\\y\end{bmatrix}}_{\cancel{2}\times 1}=\underbrace{\begin{bmatrix}25\\-31\end{bmatrix}}_{2\times 1}$ i.e. $\begin{bmatrix}5x+3y\\2x-7y\end{bmatrix}=\begin{bmatrix}25\\-31\end{bmatrix}$

Compare: To solve $3x=5$, multiply both sides by $3^{-1}=\frac{1}{3}$ on the left: $\cancel{3^{-1}}\cancel{3}x=3^{-1}5$ i.e. $x=\frac{5}{3}$.

To solve $Av=b$, multiply both sides on the left by $A^{-1}=\frac{1}{41}\begin{bmatrix}7&3\\2&-5\end{bmatrix}=\begin{bmatrix}\frac{7}{41}&\frac{3}{41}\\\frac{2}{41}&-\frac{5}{41}\end{bmatrix}$

$$Av=b$$
$$A^{-1}Av=A^{-1}b$$

$$\frac{1}{41}\begin{bmatrix}7&3\\2&-5\end{bmatrix}\begin{bmatrix}5&3\\2&-7\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{41}\begin{bmatrix}7&3\\2&-5\end{bmatrix}\begin{bmatrix}25\\-31\end{bmatrix}$$
$$\frac{1}{41}\begin{bmatrix}41&0\\0&41\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{41}\begin{bmatrix}82\\205\end{bmatrix}$$
$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2\\5\end{bmatrix}$$