Linear Algebra

Book 3

Eg. $A = \begin{bmatrix} 1 & 3 & 0 & 41 \\ 2 & 4 & 11 & 7 \\ 0 & 3 & 0 & 4 \\ 1 & 6 & 3 & 5 \end{bmatrix}$

Expanding along the third row, $\det A = \cancel{0\begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}} - 3\begin{vmatrix} 1 & 0 & 41 \\ 2 & 11 & 7 \\ 1 & 3 & 5 \end{vmatrix} + \cancel{0\begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}} - 4\begin{vmatrix} 1 & 3 & 0 \\ 2 & 4 & 11 \\ 1 & 6 & 3 \end{vmatrix}$

$$= -3\left(\begin{vmatrix} 11 & 7 \\ 3 & 5 \end{vmatrix} + 41\begin{vmatrix} 2 & 11 \\ 1 & 3 \end{vmatrix}\right) - 4\left(\begin{vmatrix} 4 & 11 \\ 6 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 11 \\ 1 & 3 \end{vmatrix}\right)$$

$$= -3\left(55-21 + 41(6-11)\right) - 4\left(12-66 - 3(6-11)\right)$$

$$= 669.$$

(I checked this by computer.)

Wed. Nov. 8 Test. Come a few minutes early if you can.
No Quiz Fri. Nov 10, 17.
I am away Fri. Nov. 17, Mon Nov 20. Lectures for those two days will be prerecorded - check the websites.

Recall: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = ad-bc$. A is invertible iff $\det A \neq 0$, in which case $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

This formula has a generalization for $n \times n$ matrices (Cramer's Rule). This is useful although not the most computationally efficient way to compute A^{-1} if n is large.

On HW 2 you had to find A^{-1} where A is 4×4. The entries of A^{-1} have a common denominator $6 = \det A$.

Eg. $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 3 \\ 7 & 6 & 4 \end{bmatrix}$, $\det A = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 1 & 3 \\ 7 & 6 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & -3 & -7 \\ 7 & 6 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & 7 \\ 7 & 6 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & 7 \\ 0 & -8 & -31 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & 7 \\ 0 & 8 & 31 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & 7 \\ 0 & -1 & 10 \end{vmatrix}$

$$= |1| \begin{vmatrix} 3 & 7 \\ -1 & 10 \end{vmatrix} = 1 \cdot 37.$$

A^{-1} has fractional entries with common denominator 37.

Matrix of minors : $M = \begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 6 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 7 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 7 & 6 \end{vmatrix} \\ \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 7 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 7 & 6 \end{vmatrix} \\ \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -14 & -13 & 5 \\ -22 & -31 & -8 \\ 1 & -7 & -3 \end{bmatrix}$

transpose ;
apply checkerboard ;
divide by det A

$$A^{-1} = \frac{1}{37} \begin{bmatrix} -14 & 22 & 1 \\ 13 & -31 & 7 \\ 5 & 8 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{14}{37} & \frac{22}{37} & \frac{1}{37} \\ \frac{13}{37} & -\frac{31}{37} & \frac{7}{37} \\ \frac{5}{37} & \frac{8}{37} & -\frac{3}{37} \end{bmatrix}$$

Check: $A A^{-1} = \frac{1}{37} \begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 3 \\ 7 & 6 & 4 \end{bmatrix} \begin{bmatrix} -14 & 22 & 1 \\ 13 & -31 & 7 \\ 5 & 8 & -3 \end{bmatrix} = \frac{1}{37} \begin{bmatrix} 37 & 0 & 0 \\ 0 & 37 & 0 \\ 0 & 0 & 37 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

If A is a square matrix with integer entries and $\det A = \pm 1$, then A^{-1} also has integer entries.

Find a constant c such that the following matrix has determinant zero:

$$A=\begin{bmatrix}5&3&6\\1&2&4\\7&7&c\end{bmatrix}\begin{array}{l}\leftarrow u=(5\ \ 3\ \ 6)\\ \leftarrow v=(1\ \ 2\ \ 4)\\ \leftarrow w \qquad u+2v=(7\ \ 7\ \ 14)\end{array}$$

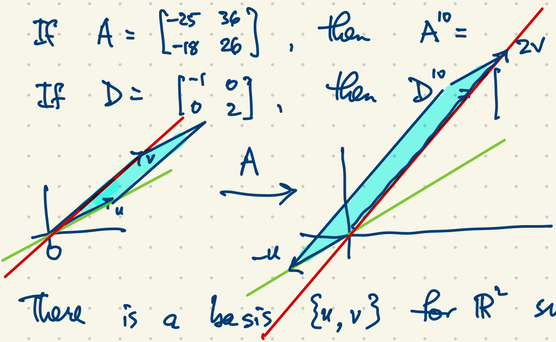
If $c=14$ then A has linearly dependent rows so $\det A = 0$ in this case (A is not invertible).

If $c \neq 14$ then A has linearly independent rows then $w \neq (7\ 7\ 14)$ and $(0\ 0\ 1)$ is a linear combination of u, v, w i.e. Row A contains $u, v, (0\ 0\ 1)$.

$$\det\left[\begin{array}{cc|c}5&3&6\\1&2&4\\ \hline 0&0&1\end{array}\right] = 7\times 1 = 7 \neq 0 .$$

If $A = \begin{bmatrix}-25&36\\-18&26\end{bmatrix}$, then $A^{10}=$

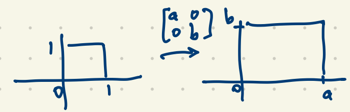
If $D = \begin{bmatrix}-1&0\\0&2\end{bmatrix}$, then $D^{10} = \begin{bmatrix} & \\ & \end{bmatrix}$



$\det A = -2$

$$\begin{vmatrix}-25&36\\-18&26\end{vmatrix} = -25\times 26 + 36\times 18 = -2$$

$$\begin{bmatrix}a&0\\0&b\end{bmatrix}\begin{bmatrix}c&0\\0&d\end{bmatrix} = \begin{bmatrix}ac&0\\0&bd\end{bmatrix}$$



Basis $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$ standard basis

$$\begin{bmatrix}x\\y\end{bmatrix} = xe_1 + ye_2$$

There is a basis $\{u, v\}$ for $\mathbb{R}^2$ such that $Au = -u$, $Av = 2v$

$$A^{10}u = AAA\cdots Au$$
$$A^2u = AAu = A(-u) = -Au = u$$
$$A^3u = AAAu = -u$$
$$\vdots$$
$$A^{10}u = u$$

$$A^2v = AAv = A(2v) = 2Av = 4v$$
$$A^3v = 8v$$
$$A^{10}v = \underbrace{1024}_{2^{10}}v$$

u, v are eigenvectors of A with corresponding eigenvalues $-1, 2$.

Definition If A is an $n \times n$ matrix, and $v \in \mathbb{R}^n$, then v is an eigenvector for A with eigenvalue λ if $Av = \lambda v$.

How do we find eigenvalues and eigenvectors?

If $Av = \lambda v$ then $Av - \lambda v = 0$ i.e. $Av - \lambda I v = 0$ i.e. $(A - \lambda I) v = 0$.
We should assume $v \neq 0$ is a nonzero null vector for $A - \lambda I$. This can only happen if $\det(A - \lambda I) = 0$.
This condition allows us to solve for the corresponding eigenvalue λ. Solve for λ; and for each value λ (each eigenvalue), solve $(A - \lambda I) v = 0$ for the corresponding eigenvector(s) v.

For $A = \begin{bmatrix} -25 & 36 \\ -18 & 26 \end{bmatrix}$, $A - \lambda I = \begin{bmatrix} -25 & 36 \\ -18 & 26 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -25-\lambda & 36 \\ -18 & 26-\lambda \end{bmatrix}$.

$$\begin{vmatrix} -25-\lambda & 36 \\ -18 & 26-\lambda \end{vmatrix} = (-25-\lambda)(26-\lambda) + 36 \cdot 18 = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2)$$

The characteristic polynomial has two roots $\lambda_1 = -1$, $\lambda_2 = 2$ (the two eigenvalues).
To find the corresponding eigenvectors v_1, v_2:
First take $\lambda_1 = -1$ and solve $Av_1 = \underbrace{-}_{\lambda_1 = -1} v_1$ i.e. $(A + I) v_1 = 0$. $A + I = \begin{bmatrix} -24 & 36 \\ -18 & 27 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (by inspection)

Or $\begin{bmatrix} -24 & 36 \\ -18 & 27 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 \\ -18 & 27 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}$ has null space $\mathrm{Span}\left\{ \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \right\}$ with basis $\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} : \quad \begin{aligned} x - \tfrac{3}{2} y &= 0 \\ 0 &= 0 \end{aligned}$$

Introduce a parameter t.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2} t \\ t \end{bmatrix} = t \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

We can take v_1 to be any nonzero scalar multiple of $\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$. I'll take $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. So $Av_1 = \lambda_1 v_1 = -v_1$.

For $\lambda_2 = 2$: Solve $Av_2 = \lambda_2 v_2 = 2v_2$ i.e. $(A-2I)v_2 = 0$ where $A - 2I = \begin{bmatrix} -25 & 36 \\ -18 & 26 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -27 & 36 \\ -18 & 24 \end{bmatrix}$

A null vector of $A - 2I$: $v_2 = \begin{bmatrix} 4/3 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. So $\begin{bmatrix} -27 & 36 \\ -18 & 24 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ i.e. $Av_2 = \lambda_2 v_2 = 2v_2$.

$v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is a basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

We started with $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as the standard basis.

To find A^{10}: two approaches.

Let $B = [v_1 \mid v_2] = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Then $AB = A\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 8 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = BD$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ (diagonal matrix)

so $A\cancel{BB^{-1}} = BDB^{-1}$ i.e. $\boxed{A = BDB^{-1}}$.

So $A^{10} = (BDB^{-1})(BDB^{-1})\cdots(BDB^{-1}) = BD^{10}B^{-1} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -8183 & 12276 \\ -6138 & 9208 \end{bmatrix}$

To check: $\det(A^{10}) = (\det A)^{10} = (-2)^{10} = 1024$. $\det\left[\ \downarrow\ \right] = 1024$.

$\det A = (-25)(26) - (36)(-18) = -2$.

$\det A = (\det B)(\det D)(\det B^{-1}) = 1 \times (-2) \times 1 = -2$

Second approach: $A^{10}v_1 = v_1$, $A^{10}v_2 = 1024 v_2$

$$\left.\begin{aligned} v_1 &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3e_1 + 2e_2 \\ v_2 &= \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 4e_1 + 3e_2 \end{aligned}\right\} \Rightarrow \begin{aligned} e_1 &= 3v_1 - 2v_2 = 3\begin{bmatrix} 3 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ e_2 &= -4v_1 + 3v_2 = -4\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

$$A^{10}e_1 = A^{10}(3v_1 - 2v_2) = 3\cdot v_1 - 2 \times 1024 v_2 = 3\begin{bmatrix} 3 \\ 2 \end{bmatrix} - 2048\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -8183 \\ -6138 \end{bmatrix}$$

$$A^{10}e_2 = A^{10}(-4v_1 + 3v_2) = -4v_1 + 3 \times 1024 v_2 = -4\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 3072\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 12276 \\ 9208 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} -8183 & 12276 \\ -6138 & 9208 \end{bmatrix}$$

A and D are similar: they represent the same linear transformation with respect to different choices of basis.