Linear Algebra

Book 2

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$f(x,y) = (3x+2y\,,\ x-5y)$ can be represented as a matrix transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{bmatrix} 3 & 2 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x+2y \\ x-5y \end{bmatrix}$$

Every linear operator can be expressed as matrix multiplication

eg. consider solutions of $y''+y=0$ i.e. $f(x) = \underbrace{a\sin x + b\cos x}_{\begin{bmatrix} a \\ b \end{bmatrix}}$

$$Df(x) = \underbrace{a\cos x - b\sin x}_{\begin{bmatrix} -b \\ a \end{bmatrix}}$$

$$D(rf+sg) = rDf + sDg$$

$$(rf+sg)' = rf' + sg'$$

$$\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{M}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$M^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$M^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Every 2×2 real matrix A represents a linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ which is the matrix transformation $T_A\begin{bmatrix}x\\y\end{bmatrix} = A\begin{bmatrix}x\\y\end{bmatrix}$.

eg. $\underbrace{\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-y\\x\end{bmatrix}$ T_A is a counter-clockwise 90° rotation about the origin in $\mathbb{R}^2$:

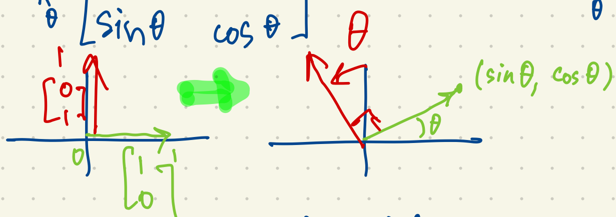
$$T_A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$T_A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$$

$$T_A^4 = I \qquad I\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

A counterclockwise rotation by angle θ about the origin in $\mathbb{R}^2$ represented by the matrix $R_\theta = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

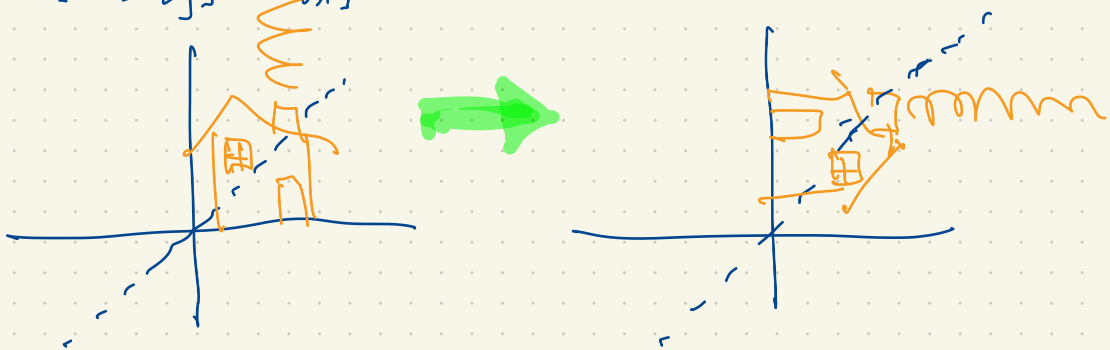
$$R_\theta\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \qquad R_\theta\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$



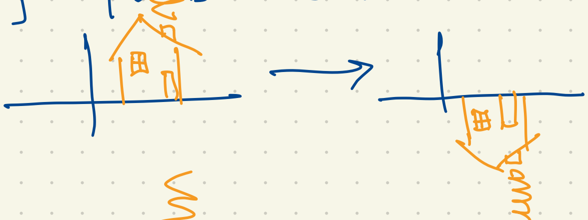
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$R_\beta R_\alpha = R_{\alpha+\beta} \qquad \begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix} = \begin{bmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{bmatrix}$$

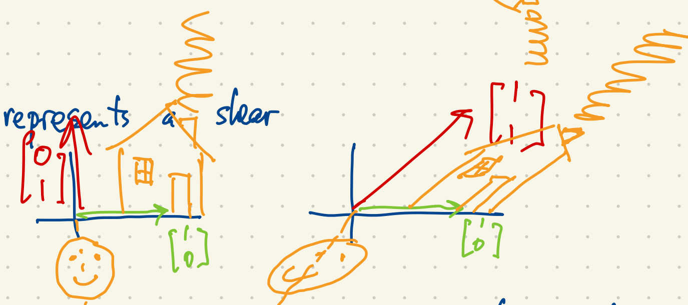
Eg. $\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}y\\x\end{bmatrix}$ is a reflection about the line $y=x$



$\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ represents a reflection in the x-axis

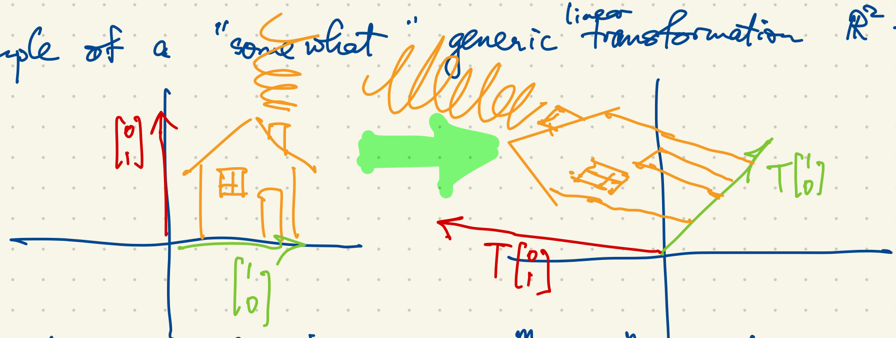


$\begin{bmatrix}1&1\\0&1\end{bmatrix}$ represents a shear



Every matrix transformation is a linear transformation: it takes 0 to 0 and it takes lines to lines or points. It may distort distances and angles.

Example of a "somewhat" generic linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$:



Every linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ takes 0 to 0,

takes lines to lines or points

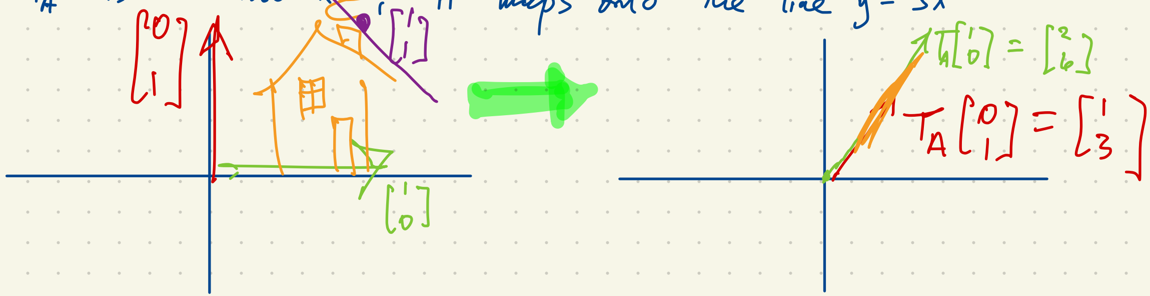
A function $f: A \to B$ is "one-to-one" if $f(x) = f(y)$ implies $x = y$. (No two inputs give the same output.)

f is "onto" if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$.

e.g. $A = \begin{bmatrix}2 & 1\\6 & 3\end{bmatrix}$ defines a linear transformation $T_A: \mathbb{R}^2 \to \mathbb{R}^2$, $T_A\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = A\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x+y\\6x+3y\end{bmatrix}$.

This function is not one-to-one e.g. $T_A\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = T_A\left(\begin{bmatrix}-1\\5\end{bmatrix}\right) = \begin{bmatrix}3\\9\end{bmatrix}$

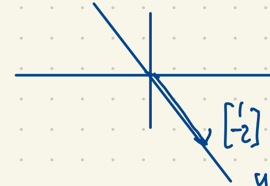
And T_A is not onto $\mathbb{R}^2$; it maps onto the line $y = 3x$



The <u>null space</u> of a linear transformation $\text{Nul}\,T = \{\underline{v} : T\underline{v} = \underline{0}\}$. (the set of null vectors of T).

Recall: $T\underline{0} = \underline{0}$

$$\text{Nul}\underbrace{\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}}_{A} = \text{Nul}\,T_A = \left\{ \begin{bmatrix} x \\ -2x \end{bmatrix} : x \in \mathbb{R} \right\}$$



$$A\begin{bmatrix} x \\ -2x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

T is one-to-one iff $\text{Nul}\,T = \{\underline{0}\}$ (the only null vector is $\underline{0}$).

This statement should be clear:

On the one hand, suppose T is one-to-one. If $\underline{v} \in \text{Nul}\,T$ then $T\underline{v} = \underline{0} = T\underline{0}$ then $\underline{v} = \underline{0}$.

This says: if T is one-to-one then $\text{Nul}\,T = \{\underline{0}\}$

Conversely, suppose $\text{Nul}\,T = \{\underline{0}\}$. If $T\underline{v} = T\underline{w}$ then $T(\underline{v}-\underline{w}) = T\underline{v} - T\underline{w} = \underline{0}$

so $\underline{v} - \underline{w} \in \text{Nul}\,T$ i.e. $\underline{v} - \underline{w} = \underline{0}$ i.e. $\underline{v} = \underline{w}$.

"<u>Span</u>" can be used as a noun or as a verb.

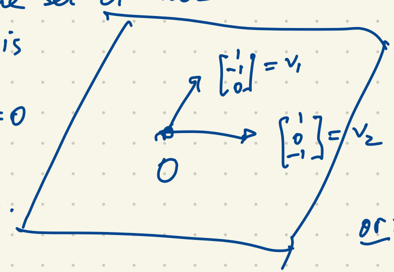
The <u>span</u> of a list of vectors $v_1, \ldots, v_k$ is the set of all linear combinations of $v_1, \ldots, v_k$.

eg. the <u>span</u> of the vectors $\underline{v_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\underline{v_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ in $\mathbb{R}^3$ is

the plane $x + y + z = 0$

in $\mathbb{R}^3$

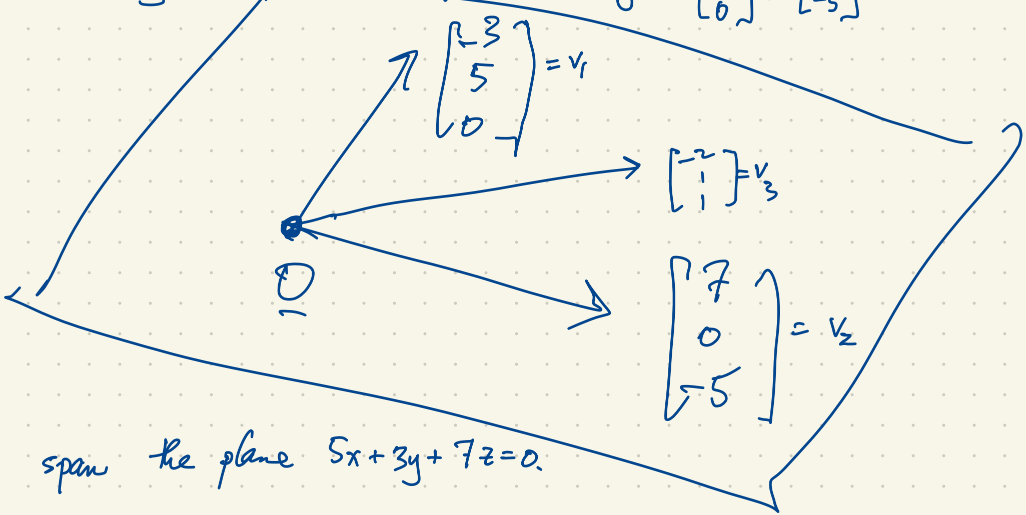
i.e. the plane $z = -x - y$.



We say that the span of v_1 and v_2 is the plane $x + y + z = 0$.

<u>or</u>: v_1 and v_2 <u>span</u> the plane $x + y + z = 0$.

eg. the plane $5x+3y+7z=0$ is spanned by $\begin{bmatrix}-3\\5\\0\end{bmatrix}, \begin{bmatrix}7\\0\\-5\end{bmatrix}$



v_1, v_2, v_3 span the plane $5x+3y+7z=0$.

Given any set of vectors $S \subset \mathbb{R}^3$, the <u>span</u> of S (denoted $\operatorname{span} S = \{$linear combinations of vectors in $S\}$) is either $\{\underline{0}\}$ or a line through $\underline{0}$, or a plane through $\underline{0}$, or $\mathbb{R}^3$.

Friday: Quiz 5 on <u>Span</u>.

The <u>image</u> of T is $\{T_A \underline{v} : \underline{v} \in$ domain of $T_A\}$ is the span of the columns of A.

Eg. $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ defines a linear transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$

(here $\mathbb{R}^3$ consists of 3×1 column vectors)

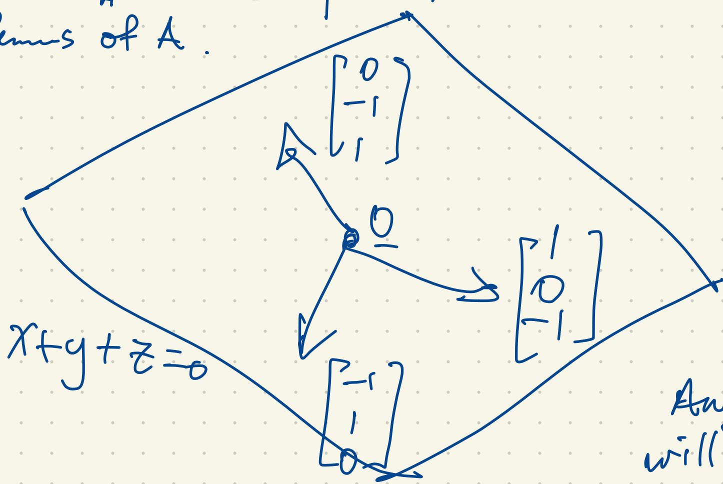
$$T_A(v) = A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y-z \\ -x+z \\ x-y \end{bmatrix}$$

$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

The image of T_A is $\{T_A v : v \in \mathbb{R}^3\} = \left\{ \underbrace{\begin{bmatrix} y-z \\ -x+z \\ x-y \end{bmatrix}}_{x\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + z\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} : x,y,z \in \mathbb{R} \right\}$

(a linear combination of the columns of A)

The image of T_A is the span of the columns of A.

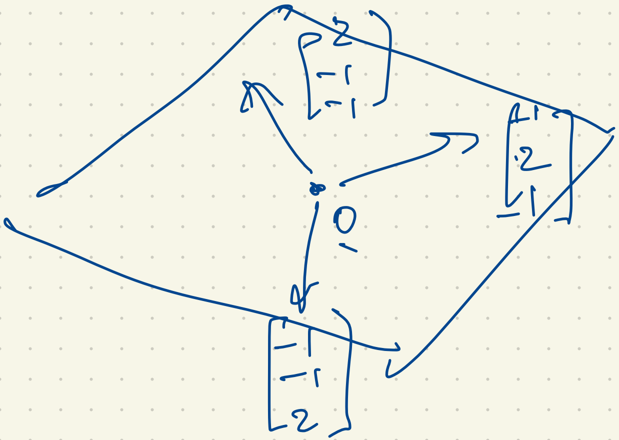


T_A is not onto $\mathbb{R}^3$. This happens because the columns of A fail to span $\mathbb{R}^3$.

Any 3 linearly independent vectors in $\mathbb{R}^3$ will span all of $\mathbb{R}^3$ (their span is $\mathbb{R}^3$).

Another example: $B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ defines a linear transformation $T_B : \mathbb{R}^3 \to \mathbb{R}^3$.

Once again T_B is not onto $\mathbb{R}^3$; its image is the span of the columns of B i.e. the plane $x+y+z=0$ through the origin in $\mathbb{R}^3$.

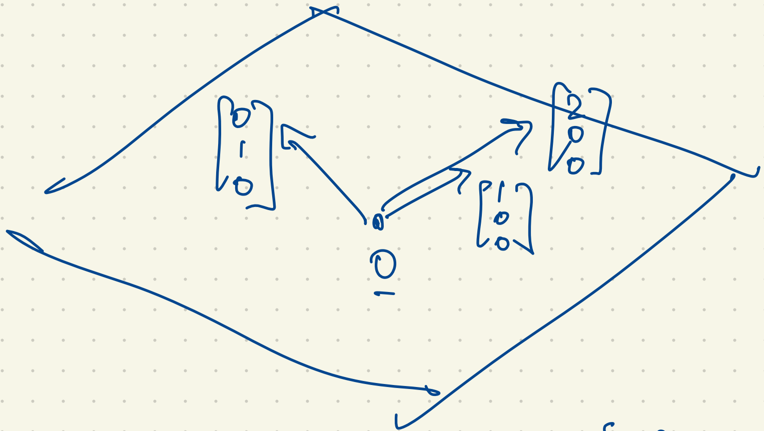


$C = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ has three linearly independent columns spanning $\mathbb{R}^3$ i.e. the image of T_C is $\mathbb{R}^3$ i.e. T_C is onto $\mathbb{R}^3$.

Check: If $a\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + c\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3a-b-c \\ -a+2b-c \\ -a-b+2c \end{bmatrix}$

$A = \begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$ has $\left\{ \begin{bmatrix}x\\ y\\ 0\end{bmatrix} : x, y \in \mathbb{R} \right\}$ as the span of its columns.

T_A is not onto.



The span of the rows of A is $\{ [a,\ 2a,\ b] : a, b \in \mathbb{R} \}$

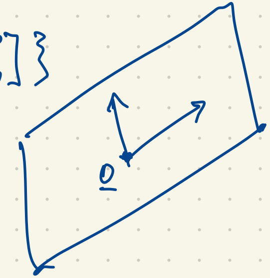
A subspace of $\mathbb{R}^n$ generalizes the notion of $\{\underline{0}\}$, line through the origin, plane through the origin, etc. up to and including $\mathbb{R}^n$ itself. The dimension of such a subspace is $0, 1, 2, 3, \dots, n$.

Given any set $S \subset \mathbb{R}^n$ (any set of vectors) then span S = {linear combinations of vectors in S} is a subspace of $\mathbb{R}^n$. Another way is to solve any homogeneous linear system in n variables. The latter case is the same thing as finding the null space of a linear transformation.

In particular if A is an $m \times n$ matrix then $\mathrm{Nul}\,A = \{ \underline{v} \in \mathbb{R}^n : A\underline{v} = \underbrace{\underline{0}}_{\text{in } \mathbb{R}^m} \}$ is a subspace of $\mathbb{R}^n$.

Eg. a 2-dimensional subspace of $\mathbb{R}^3$ (i.e. a plane through the origin) can be described in either of two ways.

$U = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \right\}$



$x + 3y - z = 0$

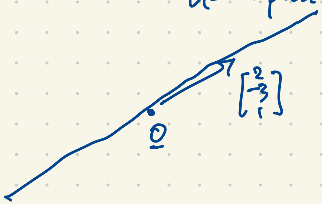
Alternatively, $U = \text{Nul}\begin{bmatrix} 1 & 3 & -1 \end{bmatrix}$

$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \right\}$

$= \left\{ s\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$

Eg. a 1-dimensional subspace of $\mathbb{R}^3$ (i.e. a line through the origin).

$U = \text{span}\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right\}$



$U = \text{Nul}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

i.e. $\begin{cases} x + y + z = 0 \\ x + 2y + 4z = 0 \end{cases}$

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$

$U = \text{Nul}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ x, y are basic variables; z is a free variable.

$z = t$ where t is arbitrary; solve for y, x

$y = -3t$

$x = 2t$

$U = \left\{ \begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\{ t\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

The solutions of $y''+y=0$ form a vector space $\{y : y''+y=0\} = \operatorname{span}\{\sin x, \cos x\}$

$= \{a \sin x + b \cos x : a, b \in \mathbb{R}\}$

Here $Ty = y''+y$ is a function mapping one function to another. $= \operatorname{Nul} T$.

$$T : \{\text{functions}\} \longrightarrow \{\text{functions}\}$$

T is a linear transformation since $T(ay_1 + by_2) = aTy_1 + bTy_2$.

Let $T: V \to W$ be a linear transformation.

T is one-to-one iff $\operatorname{Nul} T = 0$.

T is onto iff every $w \in W$ has the form $w = Tv$ for some $v \in V$.

T is bijective iff it is both one-to-one and onto. Such functions T have an inverse T^{-1}.

T^{-1} must also be linear.

Eg. consider the 2x2 matrix $A = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$ which represents a linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$.

Find the inverse matrix A^{-1}.

$A^{-1}(Av) = v$ $\qquad$ $A(A^{-1}w) = w$

$A^{-1}A = I$ $\qquad$ $AA^{-1} = I$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ identity