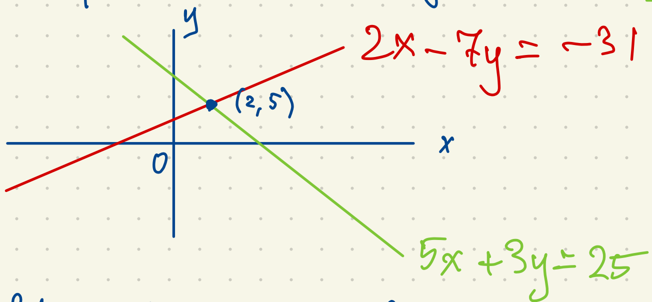
Linear Algebra

Book 1

Example: Find all (x,y) such that $5x+3y=25$ and $2x-7y=-31$.



We are asking for the simultaneous solution of a system of two equations in two unknowns x and y.

$$\begin{cases} 5x+3y=25 & (1) \\ 2x-7y=-31 & (2) \end{cases}$$

$2\times(1) - 5\times(2) = (3)$

$2\times 3 - 5(-7) = 6+35 = 41$

$2\times 25 - 5\times(-31) = 50+155 = 205$

$41y = 205$

$(4) = (3) \div 41$

$y = 5$

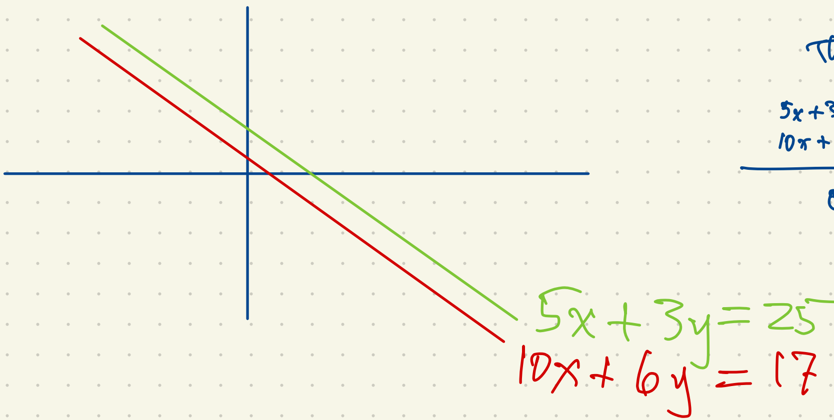
$5x + 15 = 25$

$5x = 10$

$x = 2$

Solution: $(x,y) = (2,5)$ is the unique solution.

Example: Find all (x,y) such that $5x+3y=25$ and $10x+6y=17$.



This system is inconsistent: it has no solution.

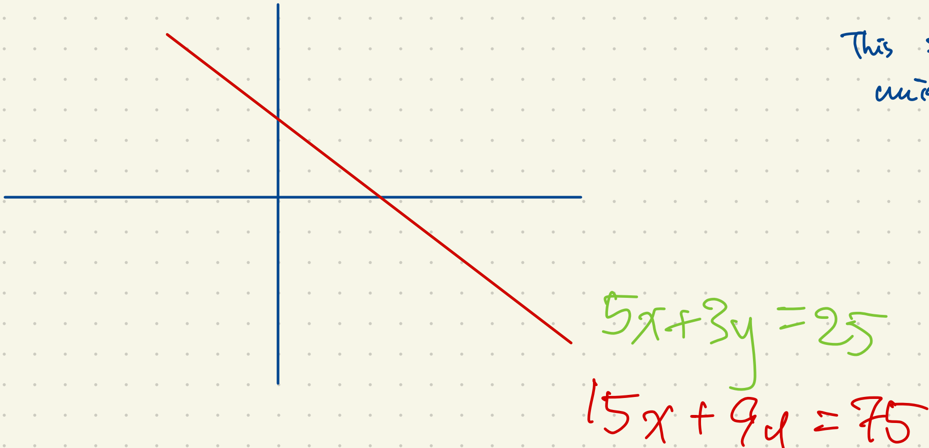
$5x+3y=25$ (1)

$10x+6y=17$ (2)

$0 = 33$ $\quad 2\times(1) - (2)$

This is inconsistent.

<u>Example</u> : Find all (x,y) such that $5x+3y=25$ and $15x+9y=75$.



This system is consistent but the solution is not unique: there are infinitely many solutions.

$$\begin{array}{rll} 5x+3y=25 & & (1) \\ 15x+9y=75 & & (2) \\ \hline 0=0 & & (3) = 3\times(1)-(2) \end{array}$$

A system of m linear equations in n unknowns has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

(a_{ij}, b_i constants for $i \in \{1,\dots,m\}$, $j = \{1,2,\dots,n\}$; $x_1,\dots,x_n$ variables representing unknowns).

<u>Typically</u>, when $m=n$ we can expect a unique solution;
$m>n$. - - - : no solution (inconsistent system);
$m<n$. — - : more than one solution.

Example with $m=n=3$: a system of 3 linear equations in 3 unknowns.

Kim buys a bag of 26 items weighing 226 oz. costing \$34. The items included

cans of tuna (\$1 each, 5 oz each)
apples (\$1 each, 8 oz each)
loaves of bread (\$3 each, 20 oz each)

How many of each item did Kim buy? (say x cans of tuna, y apples, z loaves of bread)

$$x + y + z = 26 \qquad (1)$$
$$5x + 8y + 20z = 226 \qquad (2)$$
$$x + y + 3z = 34 \qquad (3)$$

$$2z = 8 \qquad (3)-(1) = (4)$$
$$z = 4 \qquad (5)$$
$$x + y = 22 \qquad (6) = (8)-(5)$$
$$5x + 8y = 146 \qquad (7)$$
$$3y = 36 \qquad (7) - 5\times(6) = (8)$$
$$y = 12 \qquad (9) = (8)\div 3$$
$$x = 10 \qquad (10) = (6)-(9).$$

$146 - 5\times 22 = 146 - 110 = 36$

The unique solution of this system is $(x,y,z) = (10,12,4)$. (Kim bought 10 cans of tuna, 12 apples, and 4 loaves of bread.)

Check: that all three equations are satisfied.

Matrix formulation of linear systems

$$\begin{aligned} x + y + z &= 26 \\ 5x + 8y + 20z &= 226 \\ x + y + 3z &= 34 \end{aligned} \longrightarrow \begin{array}{cccc} x & y & z & \text{total} \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 1 & 1 & 3 & 34 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 1 & 1 & 3 & 34 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 0 & 0 & 2 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 0 & 3 & 15 & 96 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 0 & 1 & 5 & 32 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

subtract row 1 from row 3; divide row 3 by 2; subtract 5 times row 1 from row 2; divide row 2 by 3

$$226 - 5 \times 26 = 226 - 130 = 96$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 14 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{array}\right] \quad \text{i.e.} \quad \begin{aligned} x &= 10 \\ y &= 12 \\ z &= 4 \end{aligned}$$

subtract 5 times row 3 from row 2; subtract row 2 from row 1; subtract row 3 from row 1

Example: Find all (x, y) such that $5x + 3y = 25$ and $2x - 7y = -31$.

$$\begin{array}{cc} x & y \end{array}$$
$$\left[\begin{array}{cc|c} 5 & 3 & 25 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3/5 & 5 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{3}{5} & 5 \\ 0 & -\frac{41}{5} & -41 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{3}{5} & 5 \\ 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$$

divide row 1 by 5; subtract 2 times row 1 from row 2; multiply row 2 by $-\frac{5}{41}$; subtract $\frac{3}{5}$ times row 2 from row 1

$$-7 - \frac{6}{5} = -\frac{35}{5} - \frac{6}{5} = -\frac{41}{5}$$

$$-31 - 10 = -41$$

Solution: $(x, y) = (2, 5)$.

Alternatively:

$$\left[\begin{array}{cc|c} 5 & 3 & 25 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -7 & -31 \\ 5 & 3 & 25 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{7}{2} & -\frac{31}{2} \\ 5 & 3 & 25 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$$

interchange rows 1 and 2

Even better:

$$\left[\begin{array}{cc|c} 5 & 3 & 25 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 17 & 87 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 17 & 87 \\ 0 & -41 & -205 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 17 & 87 \\ 0 & 1 & 5 \end{array}\right]$$

subtract 2 times row 2 from row 1

subtract 2 times row 1 from row 2

divide row 2 by -41

$$-31 - 2\times 87 = -31 - 174 = -205$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$$

subtract 17 times row 2 from row 1

Solution: $(x, y) = (2, 5)$.

Check! $5\times 2 + 3\times 5 = 25$

$2\times 2 - 7\times 5 = -31$

Elementary row operations:

(i) add a multiple of one row to another

(ii) multiply a row by a nonzero constant

(iii) interchange two rows

$A \sim B$ means that A, B are linear systems having the same solutions.

We use <u>Gaussian</u> <u>elimination</u> to reduce $A_1 \sim A_2 \sim \dots \sim A_m$ where A_1 represents the linear system and A_m represent an equivalent linear system (ie. having the same solutions) but A_m is simpler than A_1. Each step $A_i \sim A_{i+1}$ is obtained by one elementary row operation.

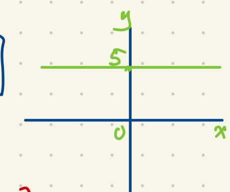
why just one operation at a time?

$$\left.\begin{array}{l} 5x + 3y = 25 \\ 2x - 7y = -31 \end{array}\right\} \quad \left[\begin{array}{cc|c} 5 & 3 & 25 \\ 2 & -7 & -31 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{3}{5} & 5 \\ 1 & -\frac{7}{2} & -\frac{31}{2} \end{array}\right] \not\sim \left[\begin{array}{cc|c} 0 & \frac{41}{10} & \frac{41}{2} \\ 0 & -\frac{41}{10} & -\frac{41}{2} \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 5 \\ 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

divide row 1 by 5
divide row 2 by 2

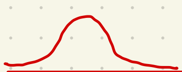
subtract row 2 from row 1
subtract row 1 from row 2

ie. $y = 5$
$0 = 0$



unique solution $(2,5)$

infinitely many solutions

Gauss  Gaussian distribution

$\left[\begin{array}{cc|c} 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$, $\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 0 & 4 \end{array}\right]$, $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$ are examples of matrices in <u>reduced row echelon form</u>:

they cannot be simplified any further by elementary row operations.

$\left[\begin{array}{cc|c} 1 & 17 & 87 \\ 0 & 1 & 5 \end{array}\right]$ is almost reduced; it is in <u>row echelon form</u>.

For a linear system whose matrix is in row echelon form, we can solve for the unknowns $x_1, x_2, \cdots, x_n$; we solve for x_n, then x_{n-1}, then x_{n-2}, $\cdots$, x_1 by <u>back-substitution</u>.

eg. $\begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 5 & 3 & 7 & 3 \\ 0 & 2 & 11 & 4 \\ 0 & 0 & 6 & 8 \end{array}\right] \end{array}$ is in row echelon form.

Every linear system has a unique reduced row echelon form.

In any $m \times n$ matrix, a <u>pivot</u> is the first nonzero entry in its row.
(Pivots are highlighted above.)

In order for a matrix to be in <u>row echelon form</u>, we must have
- pivots in any row must occur to the right of pivots in any previous rows;
- any zero rows occur at the bottom.

Assuming a matrix is already in row echelon form, then to be in <u>reduced row echelon form</u>, we must have
- every pivot entry must be a 1
- every column having a pivot has only one nonzero entry.

Example: Solve the following linear system of 3 equations in 5 unknowns:

$$\begin{cases} x_1 + 4x_2 - x_3 + 2x_4 + 3x_5 = 6 \\ 2x_1 + 8x_2 - x_3 + 7x_4 + 4x_5 = 19 \\ -x_1 - 4x_4 + 4x_3 + 8x_4 - 4x_5 = 26 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 2 & 8 & -1 & 7 & 4 & 19 \\ -1 & -4 & 4 & 8 & -4 & 26 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ -1 & -4 & 4 & 8 & -4 & 26 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 3 & 10 & -1 & 32 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right]$$

This matrix is in row echelon form. This can be used to solve the linear system by back-substitution.

$x_4 + 5x_5 = 11$ $\qquad$ $x_5 = t$ is a free parameter.

$x_4 = 11 - 5t$

$x_3 + 3x_4 - 2x_5 = 7$

$x_3 = 7 - 3x_4 + 2x_5 = 7 - 3(11-5t) + 2t = -26 + 17t$

$x_1 + 4x_2 - x_3 + 2x_4 + 3x_5 = 6$

$x_2 = s$ is another free parameter

$x_1 = 6 - 4x_2 + x_3 - 2x_4 - 3x_5 = 6 - 4s + (-26+17t) - 2(11-5t) - 3t$

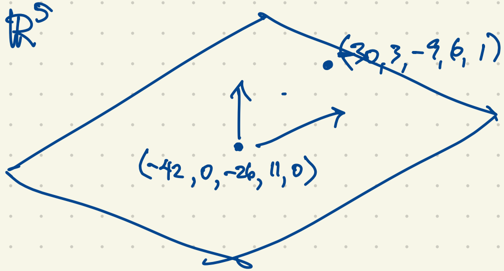
$= -42 - 4s + 24t$

Solutions: $(x_1, x_2, x_3, x_4, x_5) = (-42 - 4s + 24t,\ s,\ -26 + 17t,\ 11 - 5t,\ t)$ where s, t are arbitrary.

Geometrically, the set of solutions forms a plane (2-dimensional surface) in $\mathbb{R}^5$.

two parameters s,t are coordinates for the plane

$\mathbb{R}^5$



Solution set inside $\mathbb{R}^5$.

The (solution) point corresponding to $(s,t) = (3,1)$ is $(30, 3, -9, 6, 1)$ is another solution.

Our system is consistent but the solution is not unique.

$$\left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 5 & 1 & 13 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & 5 & 11 \end{array}\right]$$