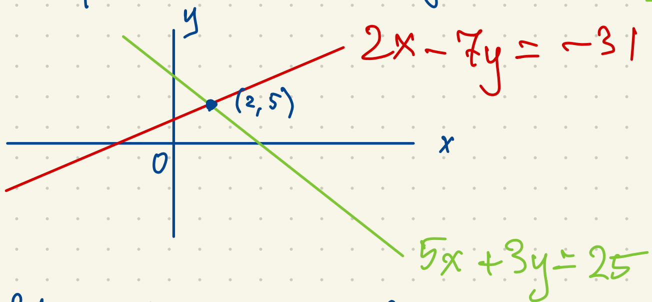
Linear Algebra

Book 1

Example: Find all (x,y) such that $5x+3y=25$ and $2x-7y=-31$.



We are asking for the simultaneous solution of a system of two equations in two unknowns x and y.

$$\begin{cases} 5x + 3y = 25 & (1) \\ 2x - 7y = -31 & (2) \end{cases}$$

$2 \times 3 - 5(-7) = 6 + 35 = 41$

$2\times(1) - 5\times(2) = (3)$

$2\times 25 - 5\times(-31) = 50 + 155 = 205$

$41y = 205$

$(4) = (3) \div 41$

$y = 5$

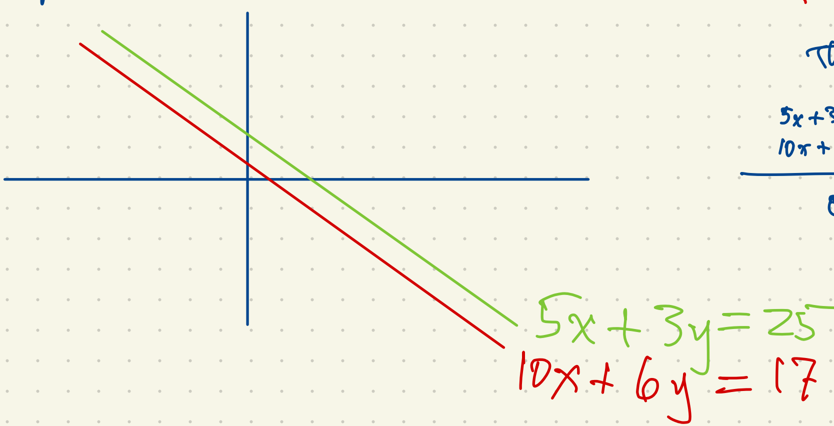
$5x + 15 = 25$

$5x = 10$

$x = 2$

Solution: $(x,y) = (2,5)$ is the unique solution.

Example: Find all (x,y) such that $5x+3y=25$ and $10x+6y=17$.



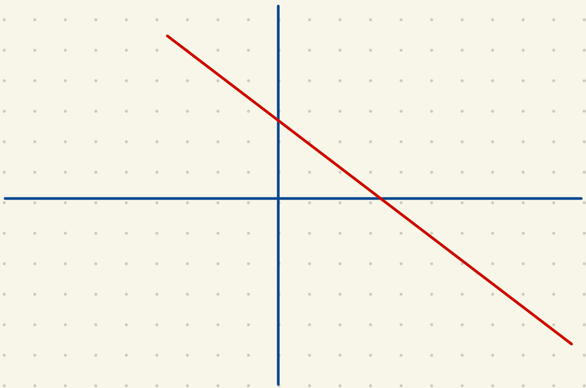
This system is inconsistent: it has no solution.

$$\begin{aligned} 5x + 3y &= 25 \quad (1) \\ 10x + 6y &= 17 \quad (2) \end{aligned}$$

$0 = 33 \qquad 2\times(1) - (2)$

This is inconsistent.

Example : Find all (x,y) such that $5x+3y=25$ and $15x+9y=75$.



$5x+3y=25$

$15x+9y=75$

This system is consistent but the solution is not unique: there are infinitely many solutions.

$$\begin{array}{rl} 5x+3y=25 & (1) \\ 15x+9y=75 & (2) \\ \hline 0=0 & (3)=3\times(1)-(2) \end{array}$$

A system of m linear equations in n unknowns has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

(a_{ij}, b_i constants for $i \in \{1,\cdots,m\}$, $j = \{1,2,\cdots,n\}$; $x_1,\cdots,x_n$ variables representing unknowns).

Typically, when $m=n$ we can expect a unique solution;
$m>n$ · - - - : no solution (inconsistent system);
$m<n$ · — — : more than one solution.

Example with $m=n=3$: a system of 3 linear equations in 3 unknowns.

Kim buys a bag of 26 items weighing 226 oz. costing \$34. The items included

cans of tuna (\$1 each, 5 oz each)
apples (\$1 each, 8 oz each)
loaves of bread (\$3 each, 20 oz each)

How many of each item did Kim buy? (say x cans of tuna, y apples, z loaves of bread)

$$
\begin{aligned}
x + y + z &= 26 && (1)\\
5x + 8y + 20z &= 226 && (2)\\
x + y + 3z &= 34 && (3)
\end{aligned}
$$

$$
\begin{aligned}
2z &= 8 && (3)-(1) = (4)\\
z &= 4 && (5)\\
x + y &= 22 && (6) = (8)-(5)\\
5x + 8y &= 146 && (7)\\
3y &= 36 && (7) - 5\times(6) = (8)\\
y &= 12 && (9) = (8)\div 3\\
x &= 10 && (10) = (6)-(9).
\end{aligned}
$$

$146 - 5\times 22 = 146 - 110 = 36$

The unique solution of this system is $(x,y,z) = (10,12,4)$. (Kim bought 10 cans of tuna, 12 apples, and 4 loaves of bread.)

Check: that all three equations are satisfied.

Matrix formulation of linear systems

$$\begin{aligned} x + y + z &= 26 \\ 5x + 8y + 20z &= 226 \\ x + y + 3z &= 34 \end{aligned} \longrightarrow \begin{array}{cccc} x & y & z & \text{total} \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 1 & 1 & 3 & 34 \end{array}\right]$$

$$226 - 5 \times 26 = 226 - 130 = 96$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 1 & 1 & 3 & 34 \end{array}\right] \underset{\text{subtract row 1 from row 3}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 0 & 0 & 2 & 8 \end{array}\right] \underset{\text{divide row 3 by 2}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 0 & 0 & 1 & 4 \end{array}\right] \underset{\text{subtract 5 times row 1 from row 2}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 0 & 3 & 15 & 96 \\ 0 & 0 & 1 & 4 \end{array}\right] \underset{\text{divide row 2 by 3}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 0 & 1 & 5 & 32 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\underset{\text{subtract 5 times row 3 from row 2}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 26 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{array}\right] \underset{\text{subtract row 2 from row 1}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 14 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{array}\right] \underset{\text{subtract row 3 from row 1}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

i.e. $x = 10$, $y = 12$, $z = 4$

Example: Find all (x, y) such that $5x + 3y = 25$ and $2x - 7y = -31$.

$$-7 - \frac{6}{5} = -\frac{35}{5} - \frac{6}{5} = -\frac{41}{5}$$

$$-31 - 10 = -41$$

$$\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 5 & 3 & 25 \\ 2 & -7 & -31 \end{array}\right] \end{array} \underset{\text{divide row 1 by 5}}{\sim} \left[\begin{array}{cc|c} 1 & 3/5 & 5 \\ 2 & -7 & -31 \end{array}\right] \underset{\text{subtract 2 times row 1 from row 2}}{\sim} \left[\begin{array}{cc|c} 1 & \frac{3}{5} & 5 \\ 0 & -\frac{41}{5} & -41 \end{array}\right] \underset{\text{multiply row 2 by } -\frac{5}{41}}{\sim} \left[\begin{array}{cc|c} 1 & \frac{3}{5} & 5 \\ 0 & 1 & 5 \end{array}\right] \underset{\text{subtract } \frac{3}{5} \text{ times row 2 from row 1}}{\sim} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$$

Solution: $(x, y) = (2, 5)$.

Alternatively:

$$\left[\begin{array}{cc|c} 5 & 3 & 25 \\ 2 & -7 & -31 \end{array}\right] \underset{\text{interchange rows 1 and 2}}{\sim} \left[\begin{array}{cc|c} 2 & -7 & -31 \\ 5 & 3 & 25 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{7}{2} & -\frac{31}{2} \\ 5 & 3 & 25 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$$

Even better: $\left[\begin{array}{cc|c}5&3&25\\2&-7&-31\end{array}\right] \sim \left[\begin{array}{cc|c}1&17&87\\2&-7&-31\end{array}\right] \sim \left[\begin{array}{cc|c}1&17&87\\0&-41&-205\end{array}\right] \sim \left[\begin{array}{cc|c}1&17&87\\0&1&5\end{array}\right]$

subtract 2 times row 2 from row 1

subtract 2 times row 1 from row 2

divide row 2 by -41

$-31-2\times 87$
$=-31-174$
$=-205$

$\sim \left[\begin{array}{cc|c}1&0&2\\0&1&5\end{array}\right]$

subtract 17 times row 2 from row 1

Solution: $(x,y)=(2,5)$.

Check! $5\times 2+3\times 5=25$
$2\times 2-7\times 5=-31$

Elementary row operations:

(i) add a multiple of one row to another
(ii) multiply a row by a nonzero constant
(iii) interchange two rows

$A\sim B$ means that A, B are linear systems having the same solutions.

We use Gaussian elimination to reduce $A_1 \sim A_2 \sim \dots \sim A_m$ where A_1 represents the linear system and A_m represents an equivalent linear system (ie. having the same solutions) but A_m is simpler than A_1. Each step $A_i \sim A_{i+1}$ is obtained by one elementary row operation.

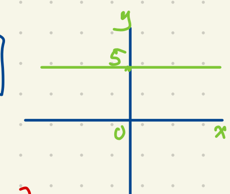
Why just one operation at a time?

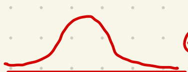
$$\left.\begin{array}{l}5x+3y=25\\2x-7y=-31\end{array}\right\} \quad \underbrace{\left[\begin{array}{cc|c}5&3&25\\2&-7&-31\end{array}\right]}_{\text{unique solution }(2,5)} \sim \left[\begin{array}{cc|c}1&\frac{3}{5}&5\\1&-\frac{7}{2}&-\frac{31}{2}\end{array}\right] \not\sim \left[\begin{array}{cc|c}0&\frac{41}{10}&\frac{41}{2}\\0&-\frac{41}{10}&-\frac{41}{2}\end{array}\right] \sim \left[\begin{array}{cc|c}0&1&5\\0&1&5\end{array}\right] \sim \left[\begin{array}{cc|c}0&1&5\\0&0&0\end{array}\right]$$

divide row 1 by 5
divide row 2 by 2

subtract row 2 from row 1
subtract row 1 from row 2

$\underbrace{\text{ie. } y=5,\ 0=0}_{\text{infinitely many solutions}}$



Gauss  Gaussian distribution

$\left[\begin{array}{cc|c} 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$, $\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 0 & 4 \end{array}\right]$, $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$ are examples of matrices in reduced row echelon form: they cannot be simplified any further by elementary row operations.

$\left[\begin{array}{cc|c} 1 & 17 & 87 \\ 0 & 1 & 5 \end{array}\right]$ is almost reduced; it is in row echelon form.

For a linear system whose matrix is in row echelon form, we can solve for the unknowns $x_1, x_2, \cdots, x_n$; we solve for x_n, then x_{n-1}, then x_{n-2}, $\cdots$, x_1 by back-substitution.

eg. $\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 5 & 3 & 7 & 3 \\ 0 & 2 & 11 & 4 \\ 0 & 0 & 6 & 8 \end{array}\right] \end{array}$ is in row echelon form.

Every linear system has a unique reduced row echelon form.

In any $m \times n$ matrix, a pivot is the first nonzero entry in its row.
(Pivots are highlighted above.)

In order for a matrix to be in row echelon form, we must have
- pivots in any row must occur to the right of pivots in any previous rows;
- any zero rows occur at the bottom.

Assuming a matrix is already in row echelon form, then to be in reduced row echelon form, we must have
- every pivot entry must be a 1
- every column having a pivot has only one nonzero entry.